Coalescent distingués échangeables et processus de Fleming-Viot généralisés avec immigration.

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A ma famille.

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Résumé

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L'objet de la thèse est d'étudier des processus stochastiques coalescents modélisant la généalogie d'une population échangeable avec immigration. On représente la population par l'ensemble des entiers $\mathbb{N} := \{1, 2, ..\}$. Imaginons que l'on échantillonne n individus dans la population aujourd'hui. On cherche à regrouper ces n individus selon leur ancêtre en remontant dans le temps. En raison de l'immigration, il se peut qu'à partir d'une certaine génération, certains individus n'aient pas d'ancêtre dans la population. Par convention, nous les regrouperons dans un bloc que nous distinguerons en ajoutant l'entier 0. On parle du bloc distingué.

Les coalescents distingués échangeables sont des processus à valeurs dans l'espace des partitions de $\mathbb{Z}_+ := \{0, 1, 2, ...\}$. A chaque temps t est associée une partition distinguée échangeable, c'est-à-dire une partition dont la loi est invariante sous l'action des permutations laissant 0 en 0. La présence du bloc distingué implique de nouvelles coagulations, inexistantes dans les coalescents classiques. Nous déterminons un critère suffisant (et nécessaire avec conditions) pour qu'un coalescent distingué descende de l'infini. Ce qui signifie qu'immédiatement après 0, le processus n'a plus qu'un nombre fini de blocs.

D'autre part, nous nous intéresserons à une relation de dualité entre ces coalescents et des processus à valeurs dans l'espace des mesures de probabilité, appelés processus de Fleming-Viot généralisés avec immigration. Dans le cadre *simple* des coalescents distingués avec coagulations multiples non simultanées (M-coalescents), nous adaptons les flots de ponts de Bertoin-Le Gall. Cela nous permettra de définir les M-processus de Fleming-Viot avec immigration. Nous établissons ensuite des liens entre ces derniers et les processus de branchement à temps continu avec immigration. Dans le cas d'un processus de branchement avec reproduction α-stable et immigration $(\alpha - 1)$-stable, nous montrons que le processus à valeurs mesures associé, renormalisé, est un processus de Fleming-Viot avec immigration changé de temps. Enfin, nous définissons entièrement les processus de Fleming-Viot généralisés avec immigration par une approche nouvelle impliquant des flots stochastiques de partitions.

Mots-clefs

Echangeabilité, processus coalescent, processus de branchement, immigration, processus à valeurs mesures.

Distinguished exchangeable coalescent processes and generalized Fleming-Viot processes with immigration

Abstract

The purpose of the dissertation is to study stochastic coalescent processes modelling the genealogy of an exchangeable population with immigration. We represent the population by the set of integers $\mathbb{N} = \{1, 2...\}$. Suppose that we sample n individuals in the population of today. We group together individuals with the same parent at preceding generations. Due to the immigration, some individuals, from a certain generation, may have no ancestor in the population. By convention, we will gather these individuals in a block to which we add 0. We call this the distinguished block.

Processes called distinguished exchangeable coalescents are valued in the space of the partitions of $\mathbb{Z}_+ = \{0, 1, ..\}$. At each time t, we consider a distinguished exchangeable partition. That is a partition whose law is invariant under the action of permutations leaving 0 at 0. The presence of the distinguished block adds new coagulation events, which do not exist in the classic coalescent processes. We determine a sufficient condition (necessary under some hypotheses) for a distinguished coalescent to come down from infinity, meaning that immediately after 0, there are only a finite number of blocks.

On the other hand, there is a duality between these coalescent processes and some processes valued in the probability-measures space, called generalized Fleming-Viot processes with immigration. In the framework of the distinguished coalescent with multiple not simultaneous coagulations (the so-called M-coalescents), we adapt the theory of flow of bridges due to Bertoin and Le Gall. This allows us to define the M-generalized Fleming-Viot process with immigration. We then establish some links between them and the continuous branching processes with immigration. In the case of a process with α-stable reproduction and $(\alpha - 1)$-stable immigration, we show that the corresponding measure-valued process, properly renormalized, is a time-changed Fleming-Viot process with immigration. Finally, we define entirely the generalized Fleming-Viot processes with immigration by introducing the new concept of stochastic flows of partitions.

Keywords

Coalescent process, exchangeability, random partition, measure-valued process, branching process with immigration

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Introduction

Ce document présente les travaux de recherche accomplis sous la direction de Jean Bertoin durant trois années de thèse. Les trois premiers chapitres correspondent à des articles soumis à des revues à comité de lecture. Le premier est constitué par l'article intitulé "distinguished exchangeable coalescents and generalized Fleming-Viot processes with immigration" publié dans *Advances in Applied Probability* vol 43. (2), [Fou11]. Dans ce chapitre, nous définissons les coalescents distingués, ainsi que les M-processus de Fleming-Viot généralisés avec immigration (M-GFVIs), processus à valeurs dans l'espace des mesures de probabilités. Pour définir ces derniers nous adaptons la théorie des flots de ponts de Bertoin-Le Gall. Contrairement aux ponts classiques, les ponts considérés auront des sauts en 0 qui représentent l'immigration. Le second chapitre correspond à l'article écrit en collaboration avec Olivier Hénard, intitulé "Stable-continuous branching processes with immigration and Beta-Fleming-Viot processes with immigration", [FH]. Nous déterminons les processus de branchement avec immigration à valeurs mesures dont le processus ratio, changé de temps, est un processus de Fleming-Viot avec immigration. Les processus de branchement avec immigration stables y jouent un rôle central. Dans le troisième chapitre nous définissons en toute généralité les processus de Fleming-Viot avec immigration en introduisant le concept des *flots stochastiques de partitions* (définis indépendamment par Cyril Labbé dans [Laba]). Nous nous intéressons au phénomène d'extinction des types initiaux. Ce chapitre est basé sur l'article "Generalized Fleming-Viot processes with immigration via stochastic flow of partitions", accepté dans *ALEA, Latin American Journal of Probability and Mathematical Statistics*, [Fou12]. Certains résultats classiques utilisés mais non démontrés dans le corps du texte, nous paraissant importants, sont rappelés et démontrés dans le dernier chapitre.

Avant de présenter les principaux résultats de cette thèse, nous donnons dans une première section quelques définitions et théorèmes classiques en théorie des populations stochastiques. Dans une deuxième section, nous aborderons les modèles avec immigration. Cela nous conduira aux principaux apports de cette thèse concernant les modèles de population échangeable de taille fixe avec immigration et leurs généalogies.

0.1 Quelques modèles classiques de populations et leurs généalogies

0.1.1 Echangeabilité et partitions échangeables

Dans les années 1920, Bruno de Finetti développa la théorie dite de l'échangeabilité. Rappelons qu'une suite de variables aléatoires $(X_n, n \geq 1)$ est dites échangeable, si sa loi est invariante sous l'action des permutations de $\mathbb{N} := \{1, 2, ...\}$ (bijections de $\mathbb{N}$ dans $\mathbb{N}$).

Le théorème suivant a été démontré par de Finetti en 1928.

Théorème (de Finetti) *Soit* $X = (X_i, i \geq 1)$ *une suite infinie de variables échangeables. Il existe une mesure de probabilité aléatoire* ρ *telle que conditionnellement à* ρ, *les variables* $(X_i, i \geq 1)$ *sont identiquement distribuées de loi* ρ. *De plus,*

$$\lim_{n\to\infty} \frac{1}{n} \sum_{j=1}^{n} \delta_{X_j} = \rho \ p.s.$$

La théorie de l'échangeabilité s'applique dans de très nombreux domaines, tout particulièrement en génétique, où sous l'impulsion de Motoo Kimura et de son étudiante de l'époque Tomoko Otha, la théorie de l'évolution neutre s'est développée (en contrepied ou complément de la selection naturelle Darwinienne). Partant du principe qu'il n'y a pas de sélection dans la population, "les individus" ont tous la même chance de se reproduire. Nous renvoyons le lecteur intéressé par ces questions au livre de John Wakeley [Wak06] et par exemple à l'article "The merits of neutral theory" de D. Alonso et al. [AEA06] qui apporte un éclairage sur la place actuelle de la théorie neutre au sein de la biologie. Afin d'illustrer nos propos, nous parlerons parfois de population infinie neutre haploïde (dernier terme qui signifie que chaque individu a un seul parent).

Plus tard, à la fin des années soixante-dix, John Kingman, [Kin78], développa la théorie des partitions aléatoires échangeables pour décrire à un temps donné les familles d'une population neutre aléatoire. Notons $\mathcal{P}_\infty$ l'espace des partitions de $\mathbb{N} := \{1, 2, ...\}$. Par convention, une partition π est représentée par la liste de ses classes d'équivalences (que nous appelerons blocs) dans l'ordre croissant de leur plus petit élément.

Définition *Une partition aléatoire échangeable est une variable aléatoire à valeurs dans l'espace* $\mathcal{P}_\infty$ *telle que pour toute permutation* σ *de* $\mathbb{N}$, *la partition* $\sigma\pi$ *définie par :*

$$i \overset{\sigma\pi}{\sim} j \Longleftrightarrow \sigma(i) \overset{\pi}{\sim} \sigma(j)$$

a la même loi que π.

Kingman établit dans [Kin78] une correspondance entre les lois des partitions échangeables et les lois des partitions de masse, à savoir les lois de probabilité sur l'espace :

$$\mathcal{P}_{\mathbf{m}} := \{\mathbf{s} = (s_1, s_2, ...); s_1 \geq s_2 \geq ... \geq 0, \sum_{i=1}^{\infty} s_i \leq 1\}.$$

L'exemple typique de partitions aléatoires est donné par ce que Kingman a appelé *paintbox* process (on parlera de boîte de peinture). Considérons un élément $s \in \mathcal{P}_{\mathbf{m}}$, on coupe l'intervalle $[0, 1]$ en sous-intervalles :

$$A_1 = [0, s_1],\ A_i = \left]\sum_{j=1}^{i-1} s_j, \sum_{j=1}^{i} s_j\right[, i \geq 1... \text{ et } \left[\sum_{j=1}^{\infty} s_i, 1\right]$$

Une $\mathbf{s}$-boîte de peinture est une partition π de $\mathbb{N}$ définie de la façon suivante. Etant donnée $(U_i)_{i\geq 1}$ une suite i.i.d de variables uniformes sur $[0, 1]$.

– Les entiers i et j sont placés dans le même bloc de π si et seulement si U_i et U_j tombent dans un même intervalle parmi $(A_1, A_2, ...)$.

– Si U_i tombe dans $[\sum_{j=1}^{\infty} s_i, 1]$, i forme un bloc singleton.

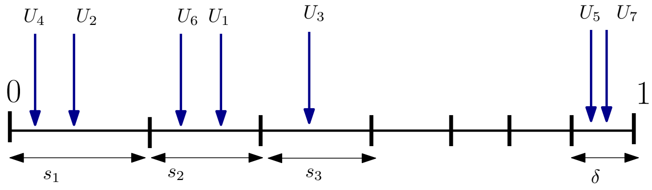


Figure 1 – Boîte de peinture

Dans cette réalisation, on a

$$\pi_{|[7]} = (\{1,6\},\{2,4\},\{3\},\{5\},\{7\}).$$

Par construction, les variables $(U_i, i \geq 1)$ étant i.i.d, la partition π est échangeable. Nous noterons sa loi $\rho_{\mathbf{s}}$, mesure de probabilité sur l'espace $\mathcal{P}_\infty$. Les positions relatives des intervalles n'ont pas d'importance, seules leurs longueurs jouent un rôle. La correspondance de Kingman s'énonce de la façon suivante :

Théorème (Kingman, 1978) *Soit π une partition aléatoire échangeable, il existe une unique mesure de probabilité ν sur $\mathcal{P}_{\boldsymbol{m}}$ telle que*

$$\mathbb{P}[\pi \in .] = \int_{\mathcal{P}_{\boldsymbol{m}}} \rho_{\boldsymbol{s}}(.)\nu(ds).$$

Reprenant le formalisme de Bertoin, Chapitre 5 de [Ber06], nous munissons l'espace $\mathcal{P}_\infty$ d'une loi de composition interne associative *coag* non commutative. Soit π et π' deux partitions, le i-ème bloc de $coag(\pi,\pi)$ est défini par :

$$coag(\pi,\pi')_i = \bigcup_{j\in\pi'_i} \pi_j.$$

La coagulation de deux partitions échangeables indépendantes est encore échangeable. Cette propriété fondamentale permet de définir des processus échangeables à valeurs partitions. En 1982, Kingman introduit le premier processus coalescent échangeable dans lequel seules des coagulations binaires ont lieu, voir [Kin82]. Ce processus est aujourd'hui au coeur de la génétique théorique. En 1999, dans deux articles indépendants, Jim Pitman, [Pit99] et Serik Sagitov, [Sag99] ont généralisé les résultats de Kingman en définissant les coalescents à collisions multiples, aussi appelés Λ-coalescents. Les coalescents échangeables généraux (à collisions multiples et simultanées), appelés Ξ-coalescents ont été finalement entièrement caractérisés en 2000 dans deux articles indépendants de Jason Schweinsberg, [Sch00a], et Möhle et Sagitov, [MS01a]. Nous définissons ces processus dans le paragraphe suivant.

0.1.2 Processus de Lévy et Processus coalescents

Nous présentons maintenant deux classes particulières de processus stochastiques, la première à valeurs réelles, bien connue est celle des processus de Lévy, la deuxième à valeurs partitions est celle des coalescents échangeables.

Un processus de Lévy (issu de 0) est un processus markovien à valeurs dans $\mathbb{R}$ vérifiant pour tous $t, s \geq 0$,

$$X_{t+s} \overset{loi}{=} X_t + \tilde{X}_s$$

où $\tilde{X}_s$ est une variable indépendante de même loi que X_s

- Ces processus sont caractérisés en loi par la formule de Lévy-Khintchine (voir Théorème 1, Chapitre 1 [Ber96]). L'exposant caractéristique η de X_t est donné par $\mathbb{E}[e^{iuX_t}] = \exp(-t\eta(u))$ avec
$$\eta(u) = ib + \frac{\sigma^2}{2}u^2 + \int_{\mathbb{R}} [1 - e^{iuy} + iuy1_{\{|y|\leq 1\}}]n(dy)$$
où n est telle que $\int_{\mathbb{R}} 1 \wedge |x^2|n(dx) < \infty$ (n est la mesure de Lévy), $b \in \mathbb{R}$ et $\sigma \geq 0$.
- Tout processus de Lévy admet la décomposition suivante (dite de Lévy-Itô) : pour tout temps t,
$$X_t = bt + \sigma B_t + \int_{0\leq|x|\leq 1} x[N(t,dx) - tn(dx)] + \int_{|x|\geq 1} xN(t,dx)$$
avec $(B_t, t \geq 0)$ un mouvement brownien, N une mesure de Poisson d'intensité $dt \otimes n$ indépendante et $N(t,dx) := \int_0^t N(ds,dx)$.

- Par définition, si X est un processus de Lévy, les variables X_t sont indéfiniment divisibles, au sens où pour tout n, il existe n variables i.i.d $Y_1, ..., Y_n$ telles que $Y_1 + ... + Y_n = X_t$. Nous renvoyons le lecteur au livre de Bertoin [Ber96] pour une étude complète des processus de Lévy.

Passons maintenant à la classe des processus coalescents et reprenons la définition du livre de Bertoin [Ber06].

Un coalescent échangeable standard est un processus $(\Pi(t), t \geq 0)$ à valeurs $\mathcal{P}_\infty$, issu de la partition singletons $0_{[\infty]} = (\{1\}, \{2\}, ...)$, avec un semi-groupe vérifiant pour tous $t, s \geq 0$,
$$\Pi(t+s) \overset{loi}{=} coag(\Pi(t), \tilde{\Pi}(s))$$
où $\tilde{\Pi}(s)$ est échangeable indépendante de même loi que $\Pi(s)$.

- Ces processus sont caractérisés en loi par la mesure de coagulation μ, mesure sur $\mathcal{P}_\infty$, (voir Théorème 4.2, Chapitre 4 de [Ber06]) admettant la décomposition
$$\mu(d\pi) = c \sum_{1\leq i<j} \delta_{K(i,j)}(d\pi) + \int_{\mathcal{P}_{\mathbf{m}}} \rho_{\mathbf{s}}(d\pi)\nu(ds),$$
où $\rho_{\mathbf{s}}$ est la loi d'une boite de peinture construite à partir de la partition de masse s, $K(i,j)$ est la partition avec pour seul bloc non singleton $\{i,j\}$. Le paramètre c est un réel positif, coefficient de coagulation binaire et ν une mesure sur $\mathcal{P}_{\mathbf{m}}$, vérifiant $\int_{\mathcal{P}_{\mathbf{m}}} \left(\sum_{i\geq 1} s_i^2\right) \nu(ds) < \infty$.

- Il n'y a pas de décomposition de type Lévy-Itô avec l'opérateur *coag*, au sens où, en général la coagulation d'un coalescent de Kingman et d'un coalescent à collisions multiples indépendants ne donne pas un coalescent.

Plaçons nous dans le cadre, dit *simple*, où la mesure μ ne charge que les partitions avec un seul bloc non-singleton (on parle de partitions simples). Ce cas correspond aux Λ-coalescents. La mesure ν figurant dans la décomposition de μ peut être vue simplement comme une mesure sur $[0,1]$. On pose alors $\Lambda(dx) = c\delta_0 + x^2\nu(dx)$, mesure finie sur $[0,1]$ et nous donnons une description heuristique, plus explicite des Λ-coalescents. Soit $\mathcal{N}$ un processus de Poisson ponctuel sur $\mathbb{R}_+ \times [0,1]$ d'intensité $dt \otimes \nu(dx)$. Posons $\Pi(0) = 0_{[\infty]}$ (partition de $\mathbb{N}$ en singletons), soit (t,x) un atome de $\mathcal{N}$, on tire une variable de Bernoulli de paramètre x sur chaque bloc présent juste avant t, tous les blocs qui ont tiré la valeur 1 coagulent. On superpose en plus de ces coagulations, des coagulations binaires à taux constant c au cours du temps. Le processus $(\Pi(t), t \geq 0)$ obtenu est un Λ-coalescent. Cette construction n'est pas toujours rigoureuse, le processus de Poisson $\mathcal{N}$ pouvant par exemple avoir des atomes qui s'accumulent. Néanmoins l'hypothèse d'intégrabilité $\int_0^1 x^2\nu(dx) < \infty$ nous permet de définir de cette façon les n-coalescents, processus à valeurs $\mathcal{P}_n$. Un argument de compatibilité permet alors de construire le processus Π à valeurs $\mathcal{P}_\infty$. Si ν est la mesure nulle, le coalescent obtenu est le coalescent de Kingman avec coefficient de coagulation c.

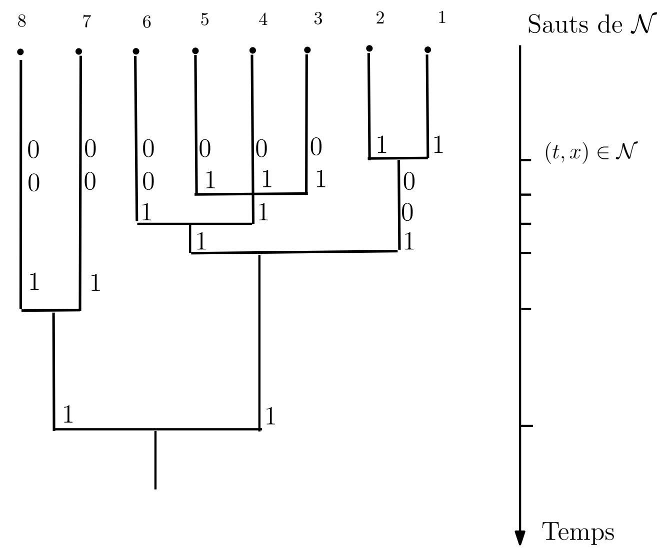


FIGURE 2 – n-Λ-coalescent

Revenons au cas général. Pitman et Schweinsberg ont mis en avant un phénomène remarquable, appelé descente de l'infini. Notons $\#\Pi(t)$ le nombre de bloc de $\Pi(t)$. Si la mesure de coagulation μ est telle que $\mu(\{\pi, \#\pi < \infty\}) = 0$ (ce qui correspond ci-dessus à faire l'hypothèse $\Lambda(\{1\}) = 0$ et implique que nous ne tirerons jamais de variable de Bernoulli avec paramètre 1), le coalescent ne peut avoir que deux sortes de comportements : presque sûrement

- Soit $\#\Pi(0) = \infty$ et $\#\Pi(t) < \infty$ quelque soit $t > 0$. On parle de descente de l'infini ;
- Soit $\#\Pi(t) = \infty$ quelque soit $t \geq 0$. Le coalescent a une infinité de blocs à tout temps.

Cette loi du 0-1 est démontrée en Annexe. Schweinsberg a déterminé des critères suffisants (et nécessaires sous des conditions de régularité) pour qu'un coalescent descende de l'infini. Nous retrouverons ces conditions dans notre cadre, un peu plus général, dans les Chapitres 1 et 3. Les arguments seront basés sur des martingales et seront différents de la preuve de Schweinsberg [Sch00b].

Les coalescents étant avant tout des objets construits afin de modéliser des généalogies de population, nous passons maintenant à la description de deux modèles de population qui ont reçu beaucoup d'attention au cours de ces cinquante dernières années.

0.1.3 Processus de branchement continu à espace d'état continu et processus de Fleming-Viot généralisés.

Nous présentons deux modèles de population continue avec des mécanismes différents. Dans le premier, la population est dite branchante et sa taille évolue. Ce processus est à valeurs dans $\mathbb{R}_+$. Son mécanisme suit la règle suivante : si initialement il y a x individus, et si nous notons $X_t(x)$ leurs descendants au temps t, alors le processus au temps $t+s$, $X_{t+s}(x)$ à même loi que $X'_s(X_t(x))$, avec $(X'_t, t \geq 0)$ indépendant et de même loi que $(X_t, t \geq 0)$. La quantité $X'_s(X_t(x))$ représente les enfants au temps s des descendants de x au temps t, et donc naturellement les descendants de x à l'instant $t+s$. Le second modèle, à taille fixe, est représenté par un processus $(\rho_t, t \geq 0)$ à valeurs dans l'espace des probabilités. Le mécanisme de reproduction pour ce second modèle suit grosso modo la règle suivante : un événement de reproduction à l'instant t correspond à tirer uniformément au hasard un individu U dans la population au temps $t-$, et à lui donner une progéniture d'une certaine taille x ; on a alors $\rho_t = (1-x)\rho_{t-} + x\delta_U$.

La référence principale utilisée ici est le cours d'Amaury Lambert [Lam08].

Population branchante et sa généalogie

L'idée de population continue en temps continu n'étant pas réellement naturelle de prime abord, nous commençons par rappeler brièvement les chaînes de Galton-Watson. Soit $(Z_n, n \geq 1)$, un processus de Galton-Watson. Par définition, Z_n est la taille de la population à la génération n, ce qui correspond à la somme des enfants des individus présents à la génération $n-1$:

$$Z_n = \sum_{i=1}^{Z_{n-1}} \xi_i$$

où les ξ_i sont indépendantes et identiquement distribuées. Réécrivons de façon un peu artificielle $Z_n - Z_{n-1} = \sum_{i=1}^{Z_n}(\xi_i - 1)$. Notons $(S_n, n \geq 1)$ la marche aléatoire de pas $Y_i = Z_i - Z_{i-1}$, on a l'identité $Z_n = S_{T_n}$ avec $T_n = \sum_{k=1}^{n} Z_k$. En d'autres termes, le processus de Galton-Watson est une marche aléatoire (avec sauts plus grands que -1) changée de temps par la taille totale de la population au temps n. Le pendant continu de la marche aléatoire $(S_n, n \geq 1)$ est un processus de Lévy spectralement positif, c'est à dire avec une mesure de Lévy portée sur $\mathbb{R}_+$. Cette analyse conduit intuitivement à la transformation de Lamperti, que nous rappelons ci après. John Lamperti démontra en 1967, voir [Lam67b], qu'une suite de chaînes de Galton-Watson, renormalisées, converge vers un processus appelé processus de branchement continu à espace d'états continu (CSBP), voir également l'article [Lam67a]. Nous rappelons les propriétés fondamentales de ces processus.

- $(X_t, t \geq 0)$ est un CSBP si et seulement si c'est un processus Markovien et si pour tous x, y fixés, $(X_t(x+y), t \geq 0) \overset{loi}{=} (X_t(x) + X'_t(y), t \geq 0)$ avec $(X'_t(y), t \geq 0)$ indépendant de même loi que $(X_t(y), t \geq 0)$. On dit qu'il vérifie la propriété de branchement. Notons que cette notion remonte à Jiřina [Jiř58].

– Transformée de Lamperti : Le CSBP $(X_t, t \geq 0)$ est un Lévy spectralement positif changé de temps : $(X_t, t \geq 0) = (Z_{U^{-1}(t \wedge T)}, t \geq 0)$ avec Z processus de Lévy d'exposant Ψ, T le temps d'atteinte de 0 de Z et U la fonctionnelle additive $U(t) := \int_0^t \frac{1}{Z_s} ds$. Cette relation n'a été rigoureusement établie que récemment, nous renvoyons le lecteur à [CLUB09], ainsi qu'à l'application du théorème de Volkonskii [Vol58] en Annexe.

– A l'instar des processus de Lévy, la loi d'un CSBP $(X_t, t \geq 0)$ est caractérisée par ses marginales unidimensionnelles (voir par exemple le théorème 1, Chapitre 2 du cours de Le Gall [LG99]) :

$$\mathbb{E}[e^{-qX_t(x)}] = \exp[-xv_t(q)], \text{ avec } \frac{\partial}{\partial t} v_t(q) = -\Psi(v_t(q)), v_0(q) = q$$

la fonction Ψ étant l'exposant de Laplace du processus de Lévy sans saut négatif codant la reproduction

$$\Psi(q) := \frac{\sigma^2}{2} q^2 + bq + \int_0^\infty (e^{-qu} - 1 + qu 1_{\{u \in (0,1)\}}) \hat{\nu_1}(du).$$

On voit facilement par la propriété de branchement que pour tout temps $t \geq 0$ fixé, le processus $(X_t(x), x \geq 0)$ est un subordinateur d'exposant de Laplace $q \mapsto v_t(q)$. Suivant les idées de Bertoin et Le Gall [BLG00], définissons un flot de subordinateurs de la façon suivante :

– Pour tous $t \geq s$, $(S^{(s,t)}(a), a \geq 0)$ est un subordinateur d'exposant de Laplace v_{t-s}.

– Pour tous temps $0 \leq t_1 \leq t_2 \leq ... \leq t_p$, $(S^{(t_i, t_{i+1})}(.), i \geq 1)$ sont des subordinateurs indépendants et
$$S^{(t_1,t_p)}(x) = S^{(t_{p-1},t_p)} \circ ... \circ S^{(t_1,t_2)}(x)$$

Les processus $(X_t(x), x \geq 0, t \geq 0)$ et $(S^{(0,t)}(x), x \geq 0, t \geq 0)$ ont les mêmes marginales fini-dimensionnelles. Le flot de subordinateurs permet de donner une notion de généalogie pour la population. L'individu a vivant à l'instant t a pour ancêtre b à l'instant s, si b est un saut de $S^{(s,t)}$ et $S^{(s,t)}(b-) < a < S^{(s,t)}(b)$. Néanmoins si cette notion de généalogie est un objet particulièrement intéressant, nous ne la donnons qu'à titre d'éclairage et renvoyons le lecteur à la Section 2 de l'article de Bertoin-Le Gall [BLG00] et au travail récent de Cyril Labbé [Labb].

Nous rappelons maintenant un résultat fondamental sur l'extinction des processus de branchement :

Théorème (Grey, [Gre74]) *Supposons qu'il existe $\theta > 0$ tel que $\Psi(z) > 0$ pour $z \geq \theta$ et que $\int_\theta^\infty \frac{dz}{\Psi(z)} < \infty$ (Conditions dites de Grey) alors $v_t(\infty) < \infty$ pour tout $t > 0$ et notant*

$$v := \downarrow \lim_{t \to \infty} v_t(\infty) \in [0, \infty) \text{ et } \zeta := \inf\{t; X_t = 0\}$$

on a
$$\mathbb{P}_x[\zeta < \infty] = \exp(-xv).$$

Le réel v est la plus grande solution de l'équation $\Psi(z) = 0$. En particulier

– *Si* $\Psi'(0) \geq 0$ *alors* $v = 0$ *ce qui implique que le processus va s'éteindre presque sûrement :* $\zeta < \infty$ *p.s.*
– *Si* $\Psi'(0) < 0$ *alors* $v > 0$.

Pour une preuve de ce théorème, voir par exemple, le théorème 3.8 de [Li11].

Population échangeable de taille fixe

En 1979, Fleming et Viot ont défini des processus permettant de modéliser le phénomène de dérive génétique (à savoir le phénomène de fixation d'un allèle simplement due aux lois du hasard). D'une certaine façon, ils ont généralisé le modèle de Wright-Fisher en considérant plusieurs "types" et en regardant des processus à valeurs mesures. Le Λ-processus de Fleming-Viot généralisé, que nous noterons $(\rho_t, t \geq 0)$, est un processus à valeurs dans l'espace des mesures de probabilité sur $[0,1]$ noté $\mathcal{M}_1$. Il est caractérisé par son générateur, noté $\mathcal{L}$ agissant sur l'ensemble des fonctions test de la forme

$$G_f : \rho \mapsto \int_{[0,1]^m} f(x_1, ..., x_m)\rho^{\otimes m}(d\mathrm{x}) \text{ par :}$$

$$\mathcal{L}G_f = c\sum_{i<j}\int_{[0,1]^m}[f(\mathrm{x}^{i,j}) - f(\mathrm{x})]\rho^{\otimes m}(d\mathrm{x}) + \int_{[0,1]} x^{-2}\Lambda(dx)\int_{[0,1]}[G_f((1-x)\rho + x\delta_u) - G_f(\rho)]\rho(du),$$

avec $\mathrm{x}_k^{i,j} = x_k$ pour tout $k \neq j$ et $\mathrm{x}_j^{i,j} = x_i$. Le premier terme correspond au Fleming-Viot classique, c'est un terme de diffusion sur l'espace des mesures. En effet, on a au sens des dérivées de Gâteaux (voir par exemple la proposition 6.5 du cours de Dawson [Daw])

$$c\sum_{i<j}\int_{[0,1]^m}[f(\mathrm{x}^{i,j}) - f(\mathrm{x})]\rho^{\otimes m}(d\mathrm{x}) = \frac{c}{2}\int_{[0,1]^2}\frac{\partial^2 G_f}{\partial\rho(x)\partial\rho(y)}[\delta_x(dy) - \rho(dy)]\rho(dx).$$

Le deuxième terme correspond à des sauts. Pour comprendre ce deuxième terme, considérons à nouveau un processus de Poisson ponctuel $\mathcal{N}$ sur $\mathbb{R}_+ \times [0,1]$ d'intensité $dt \otimes \nu(dx)$: A chaque atome (t,x) de $\mathcal{N}$, on choisit dans la population au temps $t-$, l'individu qui donnera naissance à une progéniture de taille x dans la population au temps t. Conditionnellement à ρ_{t-}, soit U une variable aléatoire de loi ρ_{t-} :

$$\rho_t = (1-x)\rho_{t-} + x\delta_U.$$

Comme précédemment pour les Λ-coalescents, dans les cas intéressants, les atomes de $\mathcal{N}$ s'accumulent. La description donnée ici n'est alors qu'heuristique. Notons que cette dynamique est fondamentalement différente du mécanisme de branchement décrit dans la section précédente. Ces processus ont été définis en 2003 par Bertoin et Le Gall qui ont généralisé la dualité entre le coalescent de Kingman et le processus de Fleming-Viot, voir [BLG03], en établissant une dualité entre les coalescents échangeables *simples* et ce qu'ils ont appelé, les processus de Fleming-Viot généralisés (appelés ici Λ-processus de Fleming-Viot). Considérons la classe de fonctions réelles définies sur $\mathcal{M}_1 \times \mathcal{P}_\infty$

$$\Phi_f(\rho, \pi) = \int_{[0,1]^m} f(x_{\alpha_\pi(1)}, ..., x_{\alpha_\pi(m)})\rho^{\otimes m}(d\mathrm{x}).$$

avec $\alpha_\pi(j) =$ indice du bloc de π contenant j. La dualité s'exprime de la façon suivante :

$$\mathbb{E}_\rho[\Phi_f(\rho_t, \pi)] = \mathbb{E}_\pi[\Phi_f(\rho, \Pi(t))],$$

où $(\Pi(t), t \geq 0)$ est un Λ-coalescent. Dans le même esprit que les flots de subordinateurs, Bertoin et Le Gall ont introduit les flots de ponts dans [BLG03]. On appelle *pont*, toute fonction aléatoire de $[0,1]$ dans $[0,1]$ de la forme

$$b_{\mathbf{s}}(r) = \sum_{i=1}^{\infty} s_i 1_{\{V_i \leq r\}} + \left(1 - \sum_{i=1}^{\infty} s_i\right) r$$

avec $\mathbf{s} \in \mathcal{P}_{\mathbf{m}}$, par définition $b_{\mathbf{s}}(0) = 0$ et $b_{\mathbf{s}}(1) = 1$. Kallenberg a démontré, voir [Kal73], qu'il s'agit en fait de la forme canonique d'une fonction de répartition d'une probabilité aléatoire échangeable sur $[0,1]$ (échangeable au sens où pour toute injection préservant la mesure de Lebesgue sur $[0,1]$, la probabilité aléatoire et son image par l'injection ont la même loi).

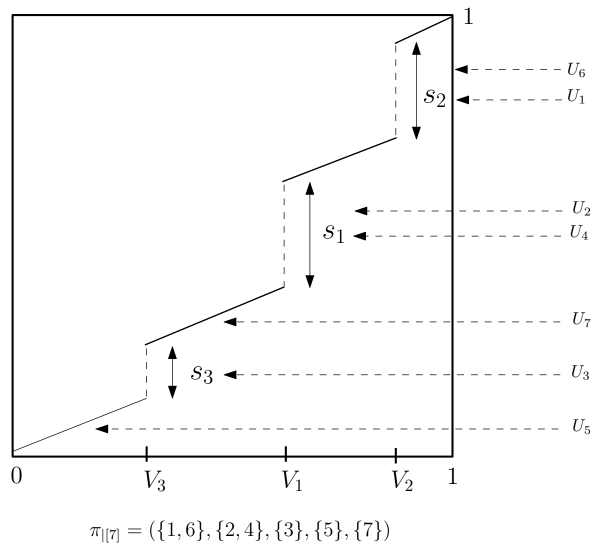


Figure 3 – pont à accroissements échangeables

Considérons une collection $(B_{s,t}, -\infty < s \leq t < \infty)$ de ponts vérifiant

- $\forall s < t < u, B_{s,u} = B_{s,t} \circ B_{t,u}$ p.s.
- La loi de $B_{s,t}$ ne dépend que de $t - s$, et pour tous $s_1 < ... < s_n$, $B_{s_1,s_2}, ..., B_{s_{n-1},s_n}$ sont indépendants.
- $B_{0,0}$ =Id, $B_{0,t} \to$ Id en probabilité lorsque $t \to 0$.

Bertoin et Le Gall ont montré qu'il existe une correspondance bijective entre les lois des coalescents échangeables et celles des flots de ponts. Soit $(U_i, i \geq 1)$ une suite de variables uniformes indépendantes. Le processus $\Pi(t)$: $i \sim j \iff B_{0,t}^{-1}(U_i) = B_{0,t}^{-1}(U_j)$ est un coalescent échangeable. Un des intérêts du flot de ponts est de pouvoir renverser le temps et ainsi considérer le flot dual $\hat{B}_{s,t} = B_{-t,-s}$ pour tous $t \geq s \geq 0$.

A partir d'un processus de Poisson ponctuel $\mathcal{N}$ sur $\mathbb{R}_+ \times [0,1]$ d'intensité $dt \otimes x^{-2}\Lambda(dx)$, Bertoin et Le Gall ont étudié un flot de ponts particulier résultant de la composition de ponts simples (i.e avec un seul saut). Ils définirent ainsi le Λ-processus de Fleming-Viot en tant que mesure de Stieltjes du pont $\hat{B}_{0,t}$, $\rho_t := d\hat{B}_{0,t}$ pour tout temps $t \geq 0$. Dans le même esprit que pour les flots de subordinateurs associés aux processus de branchement, un individu $a \in [0,1]$ est un descendant au temps t, de l'individu b vivant au temps s si $\hat{B}_{s,t}(b-) < a < \hat{B}_{s,t}(b)$. D'autre part, on peut étudier directement la généalogie de la population, en travaillant avec des processus à valeurs partitions. Soit $T > 0$, soit $(U_i, i \geq 1)$ des variables indépendantes uniformes. On définit

$$\Pi^T(t) : i \sim j \iff \hat{B}^{-1}_{T-t,T}(U_i) = \hat{B}^{-1}_{T-t,T}(U_j)$$

Le processus $(\Pi^T(t), t \in [0,T])$, représentant la généalogie de la population vivant au temps T, est un Λ-coalescent.

0.2 Modèles de population avec immigration

0.2.1 Population branchante avec immigration

En 1971, Kawazu et Watanabe ont défini dans [KW71] les processus de branchement continu avec immigration (CBI). Considérons une chaîne de Galton-Watson où en plus de la reproduction, de nouveaux individus arrivent dans la population à chaque génération n, ces individus se rajoutent aux enfants des individus de la génération $n-1$:

$$Z_n = \sum_{i=1}^{Z_{n-1}} \xi_i + I_n$$

avec $(I_n, n \geq 1)$ une suite de variables i.i.d. Passant à la limite en renormalisant, les processus obtenus forment une classe de processus appelés CBI (continuous branching process), voir par exemple la Section 3.5 du livre de Zenghu Li [Li11]. Ils sont caractérisés en loi par les deux fonctionnelles suivantes :

$$\Psi(q) = \frac{1}{2}\sigma^2 q^2 + bq + \int_0^\infty (e^{-qu} - 1 + qu)\hat{\nu_1}(du)$$
$$\Phi(q) = \beta q + \int_0^\infty (1 - e^{-qu})\hat{\nu_0}(du)$$

La première fonction est appelée mécanisme de branchement, elle caractérise la loi du Lévy spectralement positif sous-jacent. La seconde fonction est l'exposant de Laplace d'un subordinateur correspondant à l'immigration.

- $(Y_t, t \geq 0)$ est un CBI si et seulement si il est Markovien et pour tous x, y fixés, $(Y_t(x+y), t \geq 0) \overset{loi}{=} (X_t(x) + Y_t(y), t \geq 0)$, avec $(X_t(x), t \geq 0)$ un CSBP de mécanisme Ψ indépendant de $(Y_t(y), t \geq 0)$.

- La loi d'un CBI $(Y_t, t \geq 0)$ est caractérisée par ses marginales unidimensionnelles :

$$\mathbb{E}[e^{-qY_t(x)}] = \exp\left[-xv_t(q) - \int_0^t \Phi(v_s(q))ds\right], \text{ avec } \frac{\partial}{\partial t}v_t(q) = -\Psi(v_t(q)), v_0(q) = q.$$

Notons qu'il existe également une transformation de type Lamperti pour les CBIs établie par Caballero et al. dans [EPU10]. Fixons un temps t et intéressons nous au processus $(Y_t(x), x \geq 0)$. Ce processus est un subordinateur d'exposant de Laplace $q \mapsto v_t(q)$, issu d'une variable positive $Y_t(0)$ d'exposant de Laplace $\mathbb{E}[e^{-qY_t(0)}] = \exp\left(-\int_0^t \Phi(v_s(q))ds\right)$. Comme précédemment pour les CSBPs, on peut définir des flots de subordinateurs à la différence notable de la présence "d'un saut" au temps 0 dans les subordinateurs. Dans l'article [Lam02], Amaury Lambert donne une notion de généalogie pour les CBIs à travers des processus de hauteurs et des théorèmes de type Ray-Knight (objets que nous n'étudierons pas ici). Il introduit également des flots de subordinateurs associés aux CBIs (Proposition 2) et considère une famille de processus $(S^{s,t}(a), a \geq 0, s \leq t)$ sur un même espace de probabilité vérifiant les propriétés suivantes :

– Pour tous $t \geq s$, $(S^{(s,t)}(a), a \geq 0)$ est un subordinateur d'exposant de Laplace v_{t-s} issu d'une variable aléatoire de même loi que $Y_{t-s}(0)$.

– Pour tous temps $0 \leq t_1 \leq t_2 \leq ... \leq t_p$, $(S^{(t_i,t_{i+1})}(.), i \geq 1)$ sont des subordinateurs indépendants et
$$S^{(t_1,t_p)}(x) = S^{(t_{p-1},t_p)} \circ ... \circ S^{(t_1,t_2)}(x)$$

Les processus $(Y_t(x), x \geq 0, t \geq 0)$ et $(S^{(0,t)}(x), x \geq 0, t \geq 0)$ ont les mêmes marginales fini-dimensionnelles. A nouveau, le flot de subordinateurs permet de donner une notion de généalogie pour la population avec immigration. L'individu $a > 0$ vivant à l'instant t a pour ancêtre $b > 0$ à l'instant s, si b est un saut de $S^{(s,t)}$ et $S^{(s,t)}(b-) < a < S^{(s,t)}(b)$. L'individu a vivant à l'instant t est issu de l'immigration à l'instant s, si $0 < a < S^{(s,t)}(0)$.

Dans des travaux indépendants, Lambert [Lam07], Li [Li00], Roelly et Rouault [RCR89] ont démontré qu'une notion de processus de branchement continu conditionné à la non-extinction peut être définie, même lorsque l'extinction est presque sûre. Nous énonçons ici le théorème 4.1 de Lambert [Lam07], des éléments de preuve sont données en annexe (on y démontre un théorème dû à Li ne faisant pas appel aux h-transformées).

Théorème (Lambert, [Lam07]) *Soit $x > 0$, et $\mathbb{P}_x$ la loi d'un processus de branchement $(X_t, t \geq 0)$. Les lois conditionnelles $\mathbb{P}_x(.|\zeta > t)$ convergent lorsque $t \to \infty$ vers une limite notée $\mathbb{P}^{\uparrow}$ loi d'un CBI $(Y_t, t \geq 0)$ avec mécanismes Ψ pour le branchement et $\Phi(q) := \Psi'(q) - \Psi'(0+)$ pour l'immigration.*

0.2.2 Population échangeable avec immigration

Cette notion est nouvelle et fait l'objet de cette thèse. Nous donnons les principales définitions et principaux résultats. Nous commençons par définir les coalescents distingués avant de nous intéresser aux processus de Fleming-Viot généralisés avec immigration.

Coalescents distingués échangeables

Nous donnons dans cette partie quelques résultats importants du chapitre 1.

On a vu précédemment que lorsqu'une immigration est incorporée dans un processus de branchement, les subordinateurs sous-jacents ne sont pas issus de 0. Dans le contexte des populations échangeables à taille fixe, les subordinateurs sont remplacés par des ponts. On introduit donc ce qu'on appelle les ponts distingués qui "sautent" en 0. Soit $\mathbf{s}$ une partition de masse distinguée, à savoir un élément de l'espace

$$\mathcal{P}^0_{\mathbf{m}} = \{\mathbf{s} = (s_0, s_1, ...); s_0 \geq 0, s_1 \geq s_2 \geq ...; \sum_{i=1}^{\infty} s_i \leq 1\}.$$

Le pont *distingué* associé à $\mathbf{s}$ est la fonction

$$b_{\mathbf{s}}(r) = s_0 + \sum_{i=1}^{\infty} s_i 1_{\{V_i \leq r\}} + (1 - \sum_{i=0}^{\infty} s_i 1_{\{V_i \leq r\}})r$$

où les $(V_i, i \geq 1)$ sont i.i.d. uniformes sur $[0,1]$. Le pont $b_{\mathbf{s}}$ code une partition aléatoire de $\mathbb{Z}_+$, appelé boîte de peinture distinguée. Soit $(U_i, i \geq 1)$ une suite de variables indépendantes uniformes et $U_0 = 0$, on pose

$$\pi = i \sim j \Longleftrightarrow b_{\mathbf{s}}^{-1}(U_i) = b_{\mathbf{s}}^{-1}(U_j).$$

Les entiers i et j sont dans le même bloc de π si et seulement si U_i et U_j sont dans le même saut de $b_{\mathbf{s}}$.
En particulier

- Si U_i n'est pas dans un saut de $b_{\mathbf{s}}$, alors i forme un bloc singleton,
- Si U_i est dans le saut en 0 (autrement dit dans le même intervalle que U_0), alors i est dans le bloc distingué π_0.

La loi de cette partition est invariante par l'action des permutations σ de $\mathbb{Z}_+$ telle que $\sigma(0) = 0$. Le bloc π_0 est le bloc distingué et permettra de coder l'immigration dans la population.

Définition 1 *On appelle partition distingué échangeable, toute partition aléatoire de loi invariante par l'action des permutations laissant* 0 *en* 0*.*

Les lois de ces partitions sont en bijection avec les lois de probabilités sur l'espace des *partitions de masses distinguées* : $\mathcal{P}_{\mathbf{m}}^0$. Contrairement aux partitions échangeables classiques, les fréquences asymptotiques que nous avons à étudier ne sont rangées par ordre décroissant qu'à partir du deuxième terme. La correspondance de Kingman s'étend de la façon suivante :

Théorème 1 *Soit π une partition distinguée échangeable, il existe une mesure de probabilité ν sur $\mathcal{P}_{\boldsymbol{m}}^0$, telle que,*

$$\mathbb{P}[\pi \in .] = \int_{\mathcal{P}_m^0} \rho_{\boldsymbol{s}}(.) \nu(ds).$$

La mesure ν est la loi de la partition de masse distinguée de π, notée $|\pi|^{\downarrow}$ variable aléatoire à valeurs dans $\mathcal{P}_{\mathbf{m}}^0$.
Nous définissons maintenant les coalescents distingués échangeables :

Définition 2 *Un coalescent distingué échangeable est un processus de Markov* $(\Pi(t), t \geq 0)$ *avec un semi-groupe vérifiant* $\Pi(t+s) \overset{law}{=} coag(\Pi(t), \pi)$ *où π est une partition distinguée échangeable, indépendante de $\mathcal{F}_t$, avec loi ne dépendant que de s.*

A l'instar de la mesure de coagulation qui caractérise les coalescents échangeables classiques, nous obtenons une caractérisation par une mesure, que nous noterons encore μ sur $\mathcal{P}_{\infty}^0$ (appelée mesure de coagulation distinguée), invariante sous l'action des permutations σ **telles que** $\sigma(0) = 0$.

Théorème 2 *Toute mesure de coagulation distinguée admet la décomposition suivante :*

$$\mu(d\pi) = c_0 \sum_{i \geq 1} \delta_{K(0,i)}(d\pi) + c_1 \sum_{1 \leq i < j} \delta_{K(i,j)}(d\pi) + \int_{\mathcal{P}_m^0} \rho_{\boldsymbol{s}}(d\pi) \nu(ds)$$

où c_0, c_1 *sont des réels positifs,* ν *est une mesure sur* $\mathcal{P}^0_{\mathbf{m}}$*, et* $K(i,j)$ *est la partition avec pour seul bloc non trivial* $\{i,j\}$.
De plus nous avons

$$\int_{\mathcal{P}^0_m} (s_0 + \sum_{i\geq 1} s_i^2)\nu(ds) < \infty.$$

Notons que la condition d'intégrabilité vérifiée par ν diffère de celle obtenue par Schweinsberg pour les coalescents classiques (i.e non-distingués). Lorsque $c_0 = 0$ et lorsque ν est portée sur l'ensemble $\{\mathbf{s} \in \mathcal{P}^0_{\mathbf{m}}; s_0 = 0\}$ (qui s'identifie à $\mathcal{P}_{\mathbf{m}}$), le bloc distingué du coalescent est réduit au singleton $\{0\}$ (aucun individu n'est distingué), nous retrouvons les coalescents classiques.

Considérons un processus ponctuel de Poisson $\mathcal{N}$ sur $\mathbb{R}_+ \times \mathcal{P}^0_\infty$ avec intensité $dt \otimes \mu(d\pi)$. De façon heuristique, un coalescent distingué s'obtient de la façon suivante, si (t,π) est un atome de $\mathcal{N}$,

$$\Pi(t) = coag(\Pi(t-), \pi).$$

La partition π rassemble les indices des familles au temps $t-$ possédant un ancêtre commun au temps t. Les entiers dans le bloc distingué π_0 correspondent aux familles issues de l'immigration au temps t. Dans les cas intéressants, la mesure μ a une masse infinie et cette construction n'est qu'informelle à cause de l'accumulation des atomes de $\mathcal{N}$. Pour être totalement rigoureux, nous devrons dans un premier temps considérer le processus restreint à $[\,n\,] := \{0, ..., n\}$, puis utiliser un argument de compatibilité. La décomposition de μ peut s'interpréter de la façon suivante :

- le premier terme $c_0 \sum_{i\geq 1} \delta_{K(0,i)}$ peut être vu comme un terme de dérive, à taux constant c_0 un bloc vient coaguler avec le bloc distingué.
- le second terme $c_1 \sum_{1\leq i<j} \delta_{K(i,j)}$ peut être vu comme un terme de diffusion (il représente la partie Kingman)
- le dernier terme impliquant ν peut être vu comme caractérisant les "sauts" du processus.

Définition 3 *Un* M*-coalescent est un coalescent distingué échangeable simple, c'est à dire avec uniquement des coagulations multiples non simultanées. La mesure* μ *est alors portée par les partitions avec seulement un bloc non singleton.*

La mesure ν ne charge donc que l'ensemble des partitions de masse distinguées simples

$$\{\mathbf{s} \in \mathcal{P}^0_{\mathbf{m}}; \mathbf{s} = (s_0, 0, 0...) \text{ ou } \mathbf{s} = (0, s_1, 0...)\}.$$

Le symbole M correspond à une paire (Λ_0, Λ_1) de deux mesures finies sur $[0,1]$ définies de la façon suivante :

- $\Lambda_0(dx) = c_0\delta_0 + x\nu_0(dx)$ où ν_0 est la mesure sur $[0,1]$ image de ν par $\mathbf{s} \mapsto s_0$.

- $\Lambda_1(dx) = c_1\delta_0 + x^2\nu_1(dx)$ où ν_1 est la mesure sur $[0,1]$ image de ν par $\mathbf{s} \mapsto s_1$.

Nous prouvons que le (Λ_0, Λ_1)-coalescent descend de l'infini si et seulement si le Λ_1-coalescent descend. La descente de l'infini ne dépend donc pas de la mesure Λ_0.

Théorème 3 *Soit* $(\Pi(t), t \geq 0)$ *un* M*-coalescent avec* $M = (\Lambda_0, \Lambda_1)$*. Le processus descend de l'infini si et seulement si*

$$\int^{\infty} \frac{dq}{\Psi(q)} < \infty \text{ avec } \Psi(q) = c_1 q^2/2 + \int_{[0,1]} (e^{-qx} - 1 + qx) x^{-2} \Lambda_1(dx).$$

Reprenant les travaux de Bertoin-Le Gall [BLG03], [BLG05], [BLG06], nous construisons des flots de ponts "distingués", ce qui nous permet de représenter simultanément l'évolution de certaines populations appelées M-processus de Fleming-Viot avec immigration, dans le sens naturel du temps, et leurs généalogies en remontant le temps. Les généalogies sont précisément données par les M-coalescents et seront étudiées dans le chapitre 1. De façon intuitive, les M-coalescents et M-processus de Fleming-Viot peuvent être définis de la façon suivante : considérons deux processus ponctuels de Poisson $\mathcal{N}_0$ et $\mathcal{N}_1$ sur $\mathbb{R}_+ \times [0, 1]$ d'intensité respective $dt \otimes x^{-1} \Lambda_0(dx)$ et $dt \otimes x^{-2} \Lambda_1(dx)$. Nous tirons à chaque atome (t, x) de $\mathcal{N}_0$ ou $\mathcal{N}_1$ des variables de Bernoulli de paramètre x. Les blocs qui tirent 1 à un atome de temps de $\mathcal{N}_1$ coagulent, les autres restent inchangés. A chaque atome de temps de $\mathcal{N}_0$, les blocs qui tirent 1 coagulent avec la lignée 0.

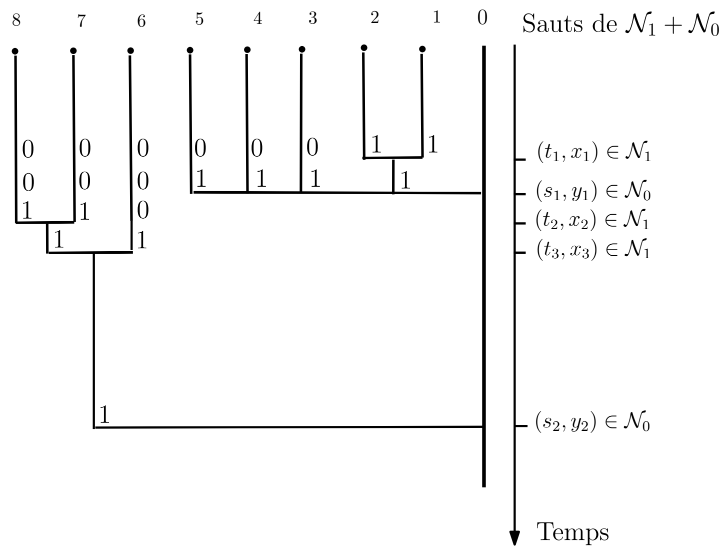


FIGURE 4 – n-M-coalescent

Une question naturelle est de savoir si nous pouvons retrouver un Λ-coalescent classique à partir d'un M-coalescent $(\Pi(t), t \geq 0)$ en ne distinguant plus les immigrés. Mathématiquement, cela correspond simplement à considérer la restriction du M-coalescent à l'ensemble des entiers $\mathbb{N}$. Nous verrons dans le chapitre 1 que le processus restreint $(\Pi_{|\mathbb{N}}(t), t \geq 0)$ peut ne pas être markovien. Le théorème suivant donne les conditions sur la paire de mesures $M = (\Lambda_0, \Lambda_1)$ pour que le M-coalescent soit un coalescent classique.

Théorème 4 *Soit* $(\Pi(t), t \geq 0)$ *un* M*-coalescent. Si* $\Lambda_0(dx) = (1 - x)\Lambda_1(dx)$ *alors* $(\Pi(t), t \geq 0)$ *est un* Λ_0*-coalescent à valeurs* $\mathcal{P}^0_\infty$.

Passons maintenant à la description du processus dual à valeurs mesures : Le M-processus de Fleming-Viot avec immigration. Ce processus décrit une population continue de *taille constante*, représentée par des types dans $[0, 1]$. Notons-le à nouveau $(\rho_t, t \geq 0)$. L'évolution de ce processus résulte de la superposition de deux mécanismes : le premier est continu dans le temps, le deuxième est discontinu. La partie continue n'est pas évidente à expliciter (et la meilleure façon de la comprendre est peut-être de regarder directement le générateur) : à taux constant c_1, on tire au hasard deux individus (i,j) et l'individu i donne son type à l'individu j. C'est la reproduction continue. En outre, au cours du temps, au taux c_0, un individu est tiré au hasard et son type remplacé par le type distingué 0. C'est l'immigration continue. Les parties discontinues sont décrites par les processus de Poisson ponctuels $\mathcal{N}_0$ et $\mathcal{N}_1$. Si (t,x) est un atome de $\mathcal{N}_0+\mathcal{N}_1$ alors t est un temps de saut du processus $(\rho_t, t\geq 0)$ et sachant ρ_{t-}, la loi conditionnelle de ρ_t est donnée par :

- $(1-x)\rho_{t-}+x\delta_U$, si (t,x) est un atome de $\mathcal{N}_1$, où U est distribuée selon ρ_{t-}
- $(1-x)\rho_{t-}+x\delta_0$, si (t,x) est un atome de $\mathcal{N}_0$.

Autrement dit, à chaque atome (t,x) de $\mathcal{N}_1$, un individu tiré dans la population au temps $t-$ donne son type à une fraction x de la population à l'instant t : c'est le même mécanisme de reproduction que pour le véritable processus de Fleming-Viot généralisé (voir [BLG03] p278). Si (t,x) est un atome de $\mathcal{N}_0$, l'individu 0 au temps $t-$ immigre en proportion x dans la population à l'instant t. On précise maintenant le générateur de ce processus. Soit f une fonction bornée sur $[0,1]^p$ et G_f la fonction-test définie comme précédemment. Notons $(\mathcal{L},\mathcal{D})$ le générateur infinitésimal de $(\rho_t, t\geq 0)$, on a :

$$\begin{aligned}\mathcal{L}G_f(\rho) &= c_1 \sum_{1\leq i<j\leq p}\int_{[0,1]^p}[f(\mathrm{x}^{i,j})-f(\mathrm{x})]\rho^{\otimes p}(d\mathrm{x})\\ &+ c_0\sum_{1\leq i\leq p}\int_{[0,1]^p}[f(\mathrm{x}^{0,i})-f(\mathrm{x})]\rho^{\otimes p}(d\mathrm{x})\\ &+\int_0^1 x^{-2}\Lambda_1(dr)\int\rho(da)[G_f((1-r)\rho+r\delta_a)-G_f(\rho)]\\ &+\int_0^1 x^{-1}\Lambda_0(dr)[G_f((1-r)\rho+r\delta_0)-G_f(\rho)].\end{aligned}$$

où x est le vecteur $(x_1,...,x_p)$ et
- le vecteur $\mathrm{x}^{0,i}$ est défini par $\mathrm{x}_k^{0,i}=x_k$, pour tout $k\neq i$ et $\mathrm{x}_i^{0,i}=0$,
- le vecteur $\mathrm{x}^{i,j}$ est défini par $\mathrm{x}_k^{i,j}=x_k$, pour tout $k\neq j$ et $\mathrm{x}_j^{i,j}=x_i$.

Par rapport au Λ-Fleming-Viot, deux termes nouveaux apparaissent. Ils sont dûs à l'immigration et sont caractérisés par la mesure finie Λ_0 avec $\Lambda_0(\{0\})=c_0$.

M-processus de Fleming-Viot et processus de branchement avec immigration stables

Nous donnons dans cette partie quelques résultats importants du chapitre 2.

Il a été établi que certains Λ-processus de Fleming-Viot peuvent être obtenus comme le ratio d'un processus branchant à valeurs mesures, changé de temps. Dans les années 90, plusieurs articles avec différentes approches ont mené à ce résultat dans le cas de la diffusion de Feller. Ce premier pont entre les modèles branchants et les modèles échangeables à taille constante a ensuite été généralisé. Quinze ans après, une connection a été établie entre les processus branchant α-stables avec $\alpha\in(0,2)$ et les $Beta(2-\alpha,\alpha)$-Fleming-Viot, où $Beta(2-\alpha,\alpha)$ correspond à la mesure finie de densité

$$f: x\mapsto x^{1-\alpha}(1-x)^{\alpha-1}1_{\{x\in]0,1]\}}.$$

Plus précisément, considérons un processus de branchement $(X_t(1), t \geq 0)$ issu de 1 avec mécanisme $\Psi(q) = q^\alpha$. Par la propriété de branchement, on peut considérer un processus à valeurs mesures sur $[0,1]$, $(M_t, t \geq 0)$ dont la masse totale est donnée par $(X_t(1), t \geq 0)$ (par exemple en prenant $M_t = dS_{0,t}$, la mesure de Stieltjes associée à $x \in [0,1] \mapsto S_{0,t}(x)$) On définit le processus ratio

$$(R_t, t \geq 0) := \left(\frac{M_t}{M_t([0,1])}, t \geq 0 \right),$$

avec temps de vie $\zeta := \inf\{t, X_t(1) = 0\}$.

Théorème (Birkner et al. [BBC$^+$05]) *Pour tout $\alpha \in (0,2]$, soit*

$$C(t) := \int_0^t \frac{ds}{X_s^{\alpha-1}(1)},$$

le processus $(R_{C^{-1}(t)}, t \geq 0)$ est un Λ-processus de Fleming-Viot généralisé avec $\Lambda = Beta(2-\alpha, \alpha)$. Le cas $\alpha = 2$ correspond au cas du branchement de Feller et a été obtenu indépendamment par Shiga, [Shi90] et Perkins [Per92], dans ce cas $\Lambda = c\delta_0$ où c est une constante.

Ces liens entre les Λ-processus de Fleming-Viot et les processus de branchement stables à valeurs mesures peuvent s'étendre dans une certaine mesure lorsqu'on considère une immigration avec de bonnes propriétés ; immigration constante pour le cas Feller, $(\alpha-1)$-stable pour le cas α-stable. A nouveau la mesure Beta joue un rôle central. Différentes méthodes sont possibles pour établir ces propriétés. Dans un travail en collaboration avec Olivier Hénard, nous nous sommes penchés sur l'étude des générateurs. L'idée fondamentale est d'étudier le couple masse totale, processus ratio et de chercher une factorisation de son générateur, en y faisant apparaître le générateur $\mathcal{L}$ du processus de Fleming-Viot avec immigration. D'après le théorème de Volkonskii, la factorisation du générateur se traduit pour le processus en terme d'un changement de temps. Nous énonçons le résultat principal du second chapitre.

Soit $\alpha \in (1,2)$, $d, d' \geq 0$, considérons les mécanismes $\Psi(q) = dq^\alpha$, $\Phi(q) = \alpha d' q^{\alpha-1}$. Soit $(Y_t(1), t \geq 0)$ un CBI issu de 1 avec mécanismes Ψ et Φ. A nouveau par la propriété de branchement, on considère $(M_t, t \geq 0)$ processus à valeurs mesures de masse totale $(Y_t(1), t \geq 0)$ et une façon de construire ce processus est de prendre la mesure de Stieltjes de $x \in [0,1] \mapsto S^{(0,t)}(x)$. La présence du saut en 0 implique une masse en 0.

Théorème 5 *Soit*

$$\left(R_t = \frac{M_t}{M_t([0,1])}, t \geq 0 \right)$$

processus avec temps de vie $\tau := \inf\{t \geq 0, Y_t(1) = 0\}$. On définit la fonctionnelle

$$C(t) := \int_0^t \frac{ds}{Y_s^{\alpha-1}(1)}.$$

Le processus changé de temps $(R_{C^{-1}(t)}, t \geq 0)$ est un M-Fleming-Viot généralisé avec

$$\Lambda_0 = d' Beta(2-\alpha, \alpha-1), \Lambda_1 = dBeta(2-\alpha, \alpha).$$

Notons que l'étude du temps de vie τ est intéressante indépendamment du résultat sur le ratio ; pour étudier le premier temps d'atteinte de 0 du processus de branchement avec immigration nous utiliserons des résultats de recouvrement dus à Fitzsimmons et al. [FFS85] (voir en annexe, théorème 4.10). Nous obtenons en particulier la dichotomie suivante : Si $\frac{d'}{d} \geq \frac{\alpha-1}{\alpha}$ alors $\mathbb{P}[\tau = \infty] = 1$, si $\frac{d'}{d} < \frac{\alpha-1}{\alpha}$ alors $\mathbb{P}[\tau < \infty] = 1$.

Lorsque l'immigration peut s'interpréter comme un conditionnement à la non-extinction, (ce qui d'après le théorème de Lambert cité précédemment équivaut à $d = d'$), le théorème permet d'établir que la généalogie du M-Fleming-Viot est un Λ-coalescent classique (vu sur $\mathcal{P}^0_\infty$).

Théorème 6 *La généalogie d'un* $(Beta(2-\alpha, \alpha-1), Beta(2-\alpha, \alpha))$*-Fleming-Viot avec immigration est un* $Beta(2-\alpha, \alpha-1)$*-coalescent.*

Processus de Fleming-Viot généralisés avec immigration

Nous donnons dans cette partie quelques résultats importants du chapitre 3. A l'instar des Ξ-processus de Fleming-Viot (définis dans [BBM$^+$09]) qui généralisent les Λ-processus de Fleming-Viot, on peut généraliser les M-processus de Fleming-Viot avec immigration.

Les M-coalescents forment une classe particulière des coalescents distingués, il en est de même pour les M-processus de Fleming-Viot avec immigration, ceux-ci peuvent être généralisés en considérant des reproductions multiples et une immigration simultanées. Nous introduirons dans le dernier chapitre le concept de flot de partitions, permettant d'étudier la population à la fois dans le sens naturel de l'évolution et dans le sens inverse du temps. L'usage de l'opérateur de coagulation dans le sens "normal" du temps est nouveau. C'est une approche naturelle pour établir la dualité entre les coalescents et les processus de Fleming-Viot qui permet de concilier la construction "lookdown" de Donnelly-Kurtz et la construction par les flots de ponts. Nous expliquons brièvement comment utiliser l'opérateur de coagulation dans ce sens. Soit $(\hat{\Pi}(t), t \geq 0)$ le processus à valeurs partitions satisfaisant pour tout $s \geq 0$,

$$\hat{\Pi}(t+s) = coag(\pi, \hat{\Pi}(s)),$$

avec π échangeable, indépendante de $\hat{\mathcal{F}}_s$ de même loi que $\hat{\Pi}(t)$. Contrairement à la définition précédente du coalescent, nous coagulons les partitions par la gauche. L'opérateur $coag$ s'interpréte alors de la façon suivante : le bloc π_j représente la descendance de l'individu j vivant à l'instant s, au temps $t+s$.

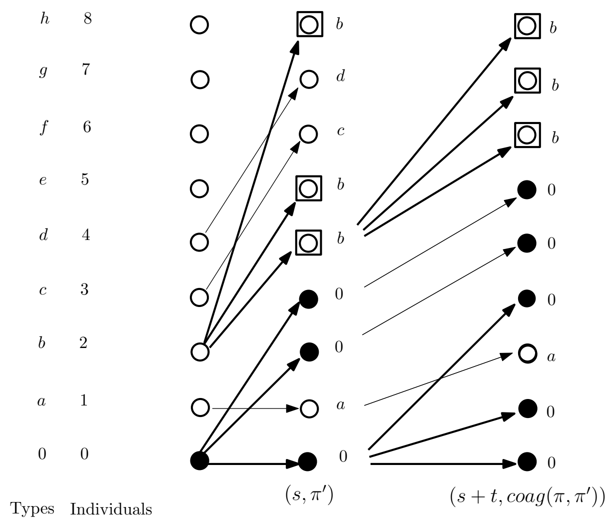
$$\begin{aligned}\hat{\Pi}_i(s+t) &= \{\text{ descendance de l'individu } i \text{ vivant au temps } 0, \text{ au temps } s+t\} \\ &= \bigcup_{j \in \hat{\Pi}_i(s)} \pi_j.\end{aligned}$$

Même s'il s'agit là d'une analogie un peu grossière, nous pouvons comparer ce mécanisme à la reproduction d'un processus de Galton-Watson, remplaçant π_j par ξ_j et la réunion par la somme. Le processus $(\hat{\Pi}(t), t \geq 0)$ représente ainsi la descendance des individus initiaux. Le bloc distingué représente quant à lui les individus issus de l'immigration.

Pour tout π, rappelons la définition de l'application α_π :

$$\alpha_\pi : k \mapsto \text{indice du bloc de } \pi \text{ contenant } k.$$

Ajoutons un type U_i à chaque individu $i \geq 1$, et fixons $U_0 = 0$. Supposons que les $(U_i, i \geq 1)$ sont indépendants de $\mathcal{N}$. La Figure 5 permet de comprendre le mécanisme de reproduc-



$\pi'_{|[8]} = (\{0,2,3\},\{1\},\{4,5,8\},\{6\},\{7\})$ descendants des individus initiaux au temps s

$\pi_{|[8]} = (\{0,1,3\},\{2\},\{4\},\{5\},\{6,7,8\})$ immigration et reproductions entre s et $s+t$

FIGURE 5 – Types evolution

tion et d'immigration. Bien que ressemblant il est différent de la construction "lookdown" de Donnelly-Kurtz, voir les articles [DK99], [BBC+05], [BBM+09]. A l'instant s, chaque individu j prend le type de son parent $\alpha_{\pi'}(j)$ vivant à l'instant 0, t générations aprés, chaque individu j vivant au temps s prend le type de son parent $\alpha_\pi(j)$. L'identité clé (voir Figure 5)

$$\alpha_{\pi'} \circ \alpha_\pi = \alpha_{coag(\pi,\pi')}$$

implique que la suite des types présents à tout instant t est donnée par $(U_{\alpha_{\hat{\Pi}(t)}(i)}, i \geq 1)$. Cette suite est échangeable, et par le théorème de de Finetti, il existe une mesure de probabilité ρ_t limite de la mesure empirique associée à la suite $(U_{\alpha_{\hat{\Pi}(t)}(i)}, i \geq 1)$:

$$\rho_t := |\hat{\Pi}_0(t)|\delta_0 + \sum_{i=1}^{\infty} |\hat{\Pi}_i(t)|\delta_{U_i} + (1 - \sum_{i=0}^{\infty} |\hat{\Pi}_i(t)|)\rho,$$

Théorème 7 *Le processus $(\rho_t, t \geq 0)$, appelé processus de Fleming-Viot généralisé avec immigration, est Fellerien et a pour générateur $\mathcal{L}$, défini sur l'espace vectoriel D engendré par les fonctionnelles du type*

$$G_f : \rho \mapsto \int_{[0,1]^p} f(x_1, ..., x_p)\rho^{\otimes p}(dx),$$

de la façon suivante :

$$\mathcal{L} = \mathcal{L}^{c_0} + \mathcal{L}^{c_1} + \mathcal{L}^{\nu}$$

où pour tout G dans D,

$$\mathcal{L}^{c_0}G(\rho) := c_0 \sum_{i\geq 1} \int_{[0,1]^m} [f(x^{0,i}) - f(x)]\rho^{\otimes m}(dx)$$

$$\mathcal{L}^{c_1}G(\rho) := c_1 \sum_{i<j} \int_{[0,1]^m} [f(x^{i,j}) - f(x)]\rho^{\otimes m}(dx)$$

$$\mathcal{L}^{\nu}G(\rho) := \int_{\mathcal{P}_m} \{\mathbb{E}[G(\bar{s}\rho + s_0\delta_0 + \sum_{i\geq 1} s_i\delta_{U_i})] - G(\rho)\}\nu(ds)$$

Les deux premiers termes correspondent respectivement à l'immigration continue et à la reproduction continue (mécanisme du processus de Fleming-Viot classique). Le troisième terme s'interprète de la façon suivante : la suite $(U_i, i \geq 1)$ est i.i.d avec loi ρ et représente les individus qui se reproduisent, chacun a pour progéniture s_i, quant à l'immigration, elle "apporte" s_0 individus dans la population. On note $\bar{s} := 1 - \sum_{i\geq 0} s_i$.

Notons que pour définir et caractériser le processus $(\rho_t, t \geq 0)$ nous introduirons des flots stochastiques de partitions. De la même façon que les flots de ponts, cela permettra d'étudier à la fois le modèle de population dans le sens "naturel" du temps, et sa généalogie en retournant le temps. Nous n'aurons pas besoin du point de vue système de particules.

En adaptant les arguments pour le M-coalescent, nous obtenons une condition suffisante sur la mesure ν (et nécessaire sous certaines conditions de régularité) pour l'extinction des types initiaux : Soit

$$\Psi(q) = c_1 q^2/2 + \int_{\mathcal{P}^0_{\mathbf{m}}} \left(\sum_{i=1}^{\infty} (e^{-qs_i} - 1 + qs_i) \right) \nu(ds)$$

Théorème 8 *Si $\mu(\pi; \pi_0 \neq \{0\}) > 0$ et si pour un certain $a > 0$, $\int_a^\infty \frac{dq}{\Psi(q)} < \infty$ alors le processus $(\rho_t, t \geq 0)$ est absorbé en temps fini en δ_0 (on parle d'extinction des types initiaux). Si de plus on a*

$$\int_{\mathcal{P}^0_m} (\sum_{i\geq 1} s_i)^2 \nu(ds) < \infty,$$

alors la condition est nécessaire.

Chapitre 1

Distinguished coalescents and M-generalized Fleming-Viot processes with immigration

This chapter is based for the most part on the article "Distinguished exchangeable coalescents and generalized Fleming-Viot processes with immigration" published in the journal Advances in Applied Probability, Volume 43-2, in june 2011. Some classic theorems used and not recalled in the chapter are stated in the Annexes. Some propositions not developed in the article are given, and notation is uniformized with the rest of the document.

1.1 Introduction

Pitman, [Pit99] and Sagitov, [Sag99] defined in 1999 the class of Λ-coalescent processes, sometimes also called simple exchangeable coalescents. These coalescent processes appear as models for the genealogy of certain haploid populations with fixed size. The general motivation of this work is to define a new class of coalescent processes that may be used to describe the genealogy of a population with immigration. Heuristically, let us imagine an infinite haploid population with immigration described at each generation by $\mathbb{N} := \{1, 2, 3, ...\}$. This means that each individual has at most one parent in the population at the previous generation ; indeed, immigration implies that some individuals may have parents outside this population (they are children of immigrants). Sampling n individuals in the population at some fixed generation, we group together the individuals with the same parent at the preceding generation. The individuals with an immigrant parent constitute a special family. We get a partition of $\mathbb{N}$ where each block is a family.
To give to the population a full genealogy, we may imagine a generic external ancestor, say 0, to distinguish the immigrants family. This way, all families will have an ancestor at the preceding generation. Following the ancestral lineage of an individual backwards in time, it may coalesce with some others in $\mathbb{N}$ or reach 0. In that last case the lineage is absorbed at 0. We call 0 the immigrant ancestor, and we shall therefore work with partitions of $\mathbb{N} \cup \{0\} = \mathbb{Z}_+$. We view the block containing 0 as distinguished and then we speak of distinguished partitions. As usual, a partition is identified with the sequence of its blocks in the increasing order of their smallest element. The distinguished block is thus the first.
For a population with no immigration, Kingman introduced exchangeable random partitions of $\mathbb{N}$. A random partition is exchangeable if and only if its law is invariant under the

action of permutations of $\mathbb{N}$. The distinguished partitions appearing in our setting are not exchangeable on $\mathbb{Z}_+$, however their laws are invariant under the action of permutations σ of $\mathbb{Z}_+$ such that $\sigma(0) = 0$. These partitions are called exchangeable distinguished partitions. We will present an extension of Kingman's theorem that determines their structure via a paint-box construction.

This allows us to define, following the approach in Bertoin's book [Ber06], a new class of coalescent processes, which we call exchangeable distinguished coalescents. An exchangeable distinguished coalescent is characterized in law by a measure μ on the space of partitions of $\mathbb{Z}_+$, called the distinguished coagulation measure. The extension of Kingman's theorem enables us to characterize this measure, and when μ is carried on the subset of simple distinguished partitions (which have only one non-trivial block), we get a representation involving two finite measures on $[0,1]$: $M = (\Lambda_0, \Lambda_1)$. We call M-coalescents this sub-class of distinguished coalescents. The restriction of an M-coalescent to each finite subset containing 0, is a Markovian coalescent chain with the following transition rates : when the partition restricted to $\mathbb{N}$ has b blocks, two kinds of jumps are allowed : for $b \geq k \geq 2$ each k-tuple of blocks not containing 0 can merge to form a single block at rate $\int_0^1 x^{k-2}(1-x)^{b-k}\Lambda_1(dx)$, and for $b \geq k \geq 1$, each k-tuple of blocks not containing 0 can merge with the one containing 0 at rate $\int_0^1 x^{k-1}(1-x)^{b-k}\Lambda_0(dx)$.

Next, we study a classical question for coalescent processes : a coalescent process starting from infinitely many blocks, is said to come down from infinity if its number of blocks instantaneously becomes finite. An interesting result is that the condition for M-coalescents to come down does not depend on Λ_0 and is the same for the Λ_1-coalescent found by Schweinsberg [Sch00b].

In the last section, we define some stochastic flows connected with M-coalescents. The model of continuous population embedded in the flow can be viewed as a generalized Fleming-Viot process with immigration. As in [BLG03], the stochastic flows involved allow us to define simultaneously a population model forward in time and its genealogical process backward in time. A duality between M-generalized Fleming-Viot processes with immigration and M-coalescents will be studied.

In a forthcoming paper, we will give a different approach to construct the generalized Fleming-Viot processes with immigration by introducing some stochastic flows of partitions. Our method will draw both on the works of Donnelly-Kurtz [DK99] and of Bertoin-Le Gall [BLG03]. Some ideas of Birkner *et al* in [BBC+05] may be applied to establish a link between certain branching processes with immigration and M-generalized Fleming-Viot processes with immigration.

Outline. The paper is organized as follows. In Section 2, we recall some basic facts on random partitions, and we give some fundamental properties of exchangeable distinguished partitions (about existence of asymptotic frequencies, paint-box representation). In Section 3, we define exchangeable distinguished coalescents. We establish a characterization of their laws by an exchangeable measure μ on the space of the distinguished partitions. The structure of μ is entirely described which enables us to study the dust. The main reference is Chapters 2 and 4 of Bertoin's book [Ber06]. The construction of exchangeable distinguished coalescents is very close to that for exchangeable coalescents of [Ber06]. In Section 4, we focus on M-coalescents and study the coming down from infinity. In particular, our approach provides a new proof of Schweinsberg's result, see [Sch00b], about necessary and sufficient conditions to come down from infinity for Λ-coalescents based on martingale arguments. In Section 5, we introduce certain stochastic flows encoding M-

coalescents. As in [Ber06] and [BLG03], these flows allow us to define a population model with immigration called M-generalized Fleming-Viot process with immigration.

1.2 Distinguished partitions

We begin with some general notation and properties which we will use constantly in the following sections.
For every integer $n \geq 1$, we denote by $[n]$ the set $\{1, ..., n\}$ and by $\mathcal{P}_n$ the set of its partitions. The set of partitions of $\mathbb{N}$ is denoted by $\mathcal{P}_\infty$. Let $\pi \in \mathcal{P}_\infty$, we identify the set π with the sequence $(\pi_1, \pi_2, ...)$ of the blocks of π enumerated in increasing order of their least element : for every $i \leq j$, $\min \pi_i \leq \min \pi_j$. The number of blocks of π is $\#\pi$. For all $\pi \in \mathcal{P}_\infty$ and $n \in \mathbb{N}$, $\pi_{|[n]} \in \mathcal{P}_n$ is by definition the restriction of π to $[n]$. We denote by $\mathcal{P}_{\mathbf{m}}$ the set of mass-partitions, meaning the decreasing sequences with sum less than or equal to 1 :

$$\mathcal{P}_{\mathbf{m}} := \{s = (s_1, s_2, ...); \sum_{i \geq 1} s_i \leq 1, s_1 \geq s_2 \geq ... \geq 0\}.$$

Given a partition $\pi = (B_1, B_2, ...)$ and a block B of that partition, we say that B has an asymptotic frequency, denoted by $|B|$, if the following limit exists :

$$|B| := \lim_{n \to \infty} \frac{\#(B \cap [n])}{n}.$$

If each block of a partition has asymptotic frequency, this partition is said to have asymptotic frequencies. For $\pi \in \mathcal{P}_\infty$ possessing asymptotic frequencies, $|\pi|^\downarrow$ is the mass partition associated with π that is $(|\pi|_i^\downarrow)_{i \in \mathbb{N}}$ is the rearrangement in decreasing order of $(|\pi_i|)_{i \in \mathbb{N}}$.
For every $n \in \mathbb{N}$, a permutation of $[n]$ is a bijection $\sigma : [n] \mapsto [n]$. For $n = \infty$, we define a permutation of $\mathbb{N}$ to be a bijection σ of $\mathbb{N}$ such that $\sigma(k) = k$ when k is large enough. We define the equivalence relation $\underset{\pi}{\sim}$ by $i \underset{\pi}{\sim} j$ if i and j are in the same block of π. We denote $\sigma\pi$ the partition defined by

$$i \underset{\sigma\pi}{\sim} j \Longleftrightarrow \sigma(i) \underset{\pi}{\sim} \sigma(j).$$

We stress that due to the ranking of the blocks, $(\sigma\pi)_i = \sigma^{-1}(\pi_{\eta(i)})$ for a certain permutation η.
A random partition π of $\mathbb{N}$ is exchangeable if $\sigma\pi$ and π have the same law, for every permutation σ of $\mathbb{N}$. Kingman established a correspondence between exchangeable partitions laws and mass-partitions via the paint-box partitions. We recall briefly the construction of paint-boxes. Let $\mathbf{s}$ be an element of $\mathcal{P}_{\mathbf{m}}$. Let $\mathcal{V}$ be an open subset of $]0, 1[$ such that the ranked sequence of lengths of its interval components is given by $\mathbf{s}$. Let $U_1, ...$ be an i.i.d sequence of uniform variables on $[0, 1]$. A $\mathbf{s}$-paint-box is the partition π induced by the equivalence relation :

$$\forall i \neq j : i \underset{\pi}{\sim} j \Longleftrightarrow U_i \text{ and } U_j \text{ belong to the same interval component of } \mathcal{V}.$$

Kingman proved that any exchangeable partition is a mixture of paint-boxes.

As explained in the Introduction, we now extend this setting by distinguishing a block, working with partitions of $\mathbb{Z}_+$.

Definition 1.1 *A distinguished partition π is a partition of $\mathbb{Z}_+$ where the block containing 0 is viewed as a distinguished block. Ranking the blocks in the order of their least element, the first block π_0 contains 0 and is the distinguished block of π.*

We denote by $[n]$ the set $\{0, 1, ..., n\}$, $\mathcal{P}_n^0$ is the space of distinguished partitions of $\{0, ..., n\}$. For $n = \infty$, we agree that $[\infty] = \mathbb{Z}_+ = \{0, 1, ...\}$ and then $\mathcal{P}_\infty^0$ is the space of partitions of $\mathbb{Z}_+$. A first basic property is the compactness of the space $\mathcal{P}_\infty^0$ for the distance defined by

$$d(\pi, \pi') = (1 + \max\{n \geq 0, \pi_{|[n]} = \pi'_{|[n]}\})^{-1}.$$

See [Ber06] for a proof. Let $\pi \in \mathcal{P}_n^0$, for all $n' \in [\infty]$ such that $n' \geq n$, we define

$$\mathcal{P}_{n',\pi}^0 = \{\pi' \in \mathcal{P}_{n'}^0; \pi'_{|[n]} = \pi\}.$$

A random distinguished partition is a random element of $\mathcal{P}_\infty^0$ equipped with the σ-field generated by the finite unions of the sets $\mathcal{P}_{n,\pi}^0$ (that corresponds to the Borelian σ-field for d).

In the same way, we introduce the set of distinguished mass-partitions, meaning the sequences $\mathbf{s} = (s_i)_{i\geq 0}$ of non-negative real numbers such that $\sum_{i\geq 0} s_i \leq 1$, ranked in decreasing order apart from s_0

$$\mathcal{P}_{\mathbf{m}}^0 := \{\mathbf{s} = (s_0, s_1, ...); \sum_{i\geq 0} s_i \leq 1, s_0 \geq 0, s_1 \geq s_2 \geq ... \geq 0\}.$$

We identify the sets $\mathcal{P}_{\mathbf{m}}$ and $\{\mathbf{s} \in \mathcal{P}_{\mathbf{m}}^0; s_0 = 0\}$. The dust of $\mathbf{s}$ is by definition the quantity $\delta := 1 - \sum_{i=0}^{\infty} s_i$. A (distinguished) mass-partition is said to be improper if the dust is positive. For $\pi \in \mathcal{P}_\infty^0$ having asymptotic frequencies, $|\pi|^{\downarrow}$ is the distinguished mass-partition associated with π that is $|\pi|_0^{\downarrow} = |\pi_0|$ and $(|\pi|_i^{\downarrow})_{i\in\mathbb{N}}$ is the rearrangement in decreasing order of $(|\pi_i|)_{i\in\mathbb{N}}$. We stress that by definition $|\pi|_0^{\downarrow} = |\pi_0|$.

We define a permutation of $\mathbb{Z}_+$ to be a bijection σ of $\mathbb{Z}_+$ such that $\sigma(k) = k$ when k is large enough. Note that any permutation of $\mathbb{N}$ can be extended to a permutation of $\mathbb{Z}_+$ by deciding that $\sigma(0) = 0$.

Definition 1.2 *A random distinguished partition π is exchangeable if $\sigma\pi$ and π have the same law for every permutation σ of $\mathbb{Z}_+$ such that $\sigma(0) = 0$.*

It is easily seen that the restriction of an exchangeable distinguished partition to $\mathbb{N}$ is exchangeable. The converse may fail : there exist distinguished partitions which are not exchangeable though their restriction to $\mathbb{N}$ is exchangeable. We construct a counter-example : let π be a non-degenerate exchangeable random partition and π' obtained from π by distinguishing the block containing 1, i.e $\pi' = (\pi_1 \cup \{0\}, \pi_2, ...)$ with blocks enumerated in order of appearance. The restriction $\pi'_{|\mathbb{N}} = \pi$ is exchangeable. The structure of π implies that $\mathbb{P}[\pi'_{|[2]} = (\{0, 1\}, \{2\})] = \mathbb{P}[1 \underset{\pi}{\not\sim} 2] > 0$. Let σ be the permutation of $[2]$: $\sigma(0) = 0, \sigma(1) = 2, \sigma(2) = 1$. We have $\mathbb{P}[\pi'_{|[2]} = (\{0, \sigma(1)\}, \{\sigma(2)\})] = \mathbb{P}[\pi_1 = \{2\}] = 0$. We thus found a permutation such that $\mathbb{P}[\pi' = \sigma(\pi'_0, ...)] \neq \mathbb{P}[\pi = (\pi_0, ...)]$.

We define now the distinguished paint-boxes and extend the Kingman's correspondence to exchangeable distinguished partitions.

Definition 1.3 *A distinguished paint-box can be constructed in the following way :*
Let $\boldsymbol{s}$ be a distinguished mass partition, we denote by δ its dust. Denote by ∂, an element which does not belong to $\mathbb{Z}_+$. Let ξ be a probability on $\mathbb{Z}_+ \cup \{\partial\}$ such that for all $k \geq 0, \xi(k) = s_k$ and $\xi(\partial) = \delta$. Drawing $X_1, X_2, ...$ a sequence of i.i.d random variables with distribution ξ and $X_0 = 0$. A $\boldsymbol{s}$-distinguished paint box is defined by : $\forall i \neq j \geq 0$,

$$\Pi : i \sim j \iff X_i = X_j \neq \partial.$$

In particular, $\Pi_0 := \{i \geq 0; X_i = 0\} = \{i \geq 1; X_i = 0\} \cup \{0\}$.

We denote by $\rho_{\mathbf{s}}$ the law of an $\mathbf{s}$-distinguished paint box. When $s_0 = 0$, the block Π_0 is the singleton $\{0\}$ and the $\mathbf{s}$-distinguished paint box restricted to $\mathbb{N}$ is a classical $\mathbf{s}$-paint box partition of $\mathbb{N}$.

Another way to define an $\mathbf{s}$-distinguished paint-box is to consider a sub-probability α on $[0,1[$ and set $\mathbf{s} = (\alpha(0), \alpha(x_1), \alpha(x_2), ...)$ where $x_1, x_2, ...$ are the atoms of α in $]0,1[$ ranked in decreasing order of their sizes. Let $X_1, X_2...$ be independent with law α and $X_0 = 0$, the partition π defined by : $i \neq j : i \sim j \iff X_i = X_j$ is an $\mathbf{s}$-distinguished paint-box.
Equivalently, we can work with uniform variables : an interval representation of $\mathbf{s}$ is a collection of disjoint intervals $(A_0, A_1, A_2, ...)$, where A_0 is $[0, s_0]$, and $(A_i)_{i\geq 1}$ such that the decreasing sequence of their lengths is $(s_1, s_2, ...)$. If we draw an infinite sequence of uniform independent variables $(U_i)_{i\geq 1}$ and fix $U_0 = 0$. The partition of $\mathbb{Z}_+$ defined by : $\pi := i \sim j$ if and only if U_i and U_j fall in the same interval (if U_i falls in the dust of $\mathbf{s}$ then $\{i\}$ is a singleton block of π) is a $\mathbf{s}$-distinguished paint box. We stress that its law does not depend on the choice of intervals $(A_1, A_2...)$ and then we can choose $A_i := [s_0 + ... + s_{i-1}, s_0 + ... + s_i[$ for all $i \geq 1$.

Proposition 1.4 *Let* $\boldsymbol{s} \in \mathcal{P}^0_{\boldsymbol{m}}$ *and* π *a* $\boldsymbol{s}$*-distinguished paint-box.*

(i) The distinguished paint-box π *is exchangeable.*

(ii) π *has asymptotic frequencies, and more precisely* $|\pi|^{\downarrow} = s$.

(iii) For every $i \in \mathbb{Z}_+$, *if* $|\pi_i| = 0$ *then* π_i *is a singleton or empty.*

(iv) $\boldsymbol{s}$ *is improper if and only if some blocks different from* π_0 *are singletons. In that case the set of singletons* $\{i \in \mathbb{Z}_+ : i \text{ is a singleton of } \pi\}$ *has an asymptotic frequency given by the dust* $\delta = 1 - \sum_{i=0}^{\infty} s_i$ *a.s.*

(v) We have $\rho_{\boldsymbol{s}}(0 \text{ is singleton }) = 0$ *if* $s_0 > 0$ *and* 1 *otherwise, and*

$$\rho_{\boldsymbol{s}}(1, 2, ..., q \text{ are singletons }) = \delta^q,$$

for $q \geq 1$.

Proof. Let σ be a permutation of $\mathbb{Z}_+$, with $\sigma(0) = 0$, $(U_i)_{i\geq 1}$ is an exchangeable sequence, and so $(U_{\sigma(i)})_{i\geq 1}$ and $(U_i)_{i\geq 1}$ have the same law. It follows that $\mathbb{P}[i \underset{\pi}{\sim} j] = \mathbb{P}[\sigma(i) \underset{\pi}{\sim} \sigma(j)]$ and then π is exchangeable. The assertion *(i)* is then established. We stress that the block $\sigma\pi_0$ is given by $\sigma^{-1}(\pi_0)$. To show *(ii)*, let $\mathcal{V} := \{[0, s_0[, A_1, A_2...\}$ be a representation of $\mathbf{s}$. Let $k \geq 0$, the block π_k is such that $\forall i, j \in \pi_k$, U_i and U_j fall in the same interval. We denote this interval by $A \in \mathcal{V}$. If $k = 0$, then $A = [0, s_0[$ because $U_0 = 0$. By the strong law of large numbers,

$$\lim_{n\to\infty} \frac{1}{n} \# \pi_k \cap [\, n\,] = \lim_{n\to\infty} \frac{1}{n} \# \{i \leq n; U_i \in A\} = |A| \text{ almost surely.}$$

We then have $|\pi|^{\downarrow} = (|\pi_0|, |\pi|^{\downarrow}_1, ...) = \mathbf{s}$. The assertion *(iii)* is plain : if $\mathbb{P}[1, 2 \in \pi_1] > 0$, then $\mathbb{P}[U_1, U_2 \in A] > 0$ for some $A \in \mathcal{V}$ and $|\pi_1| = |A| > 0$. Suppose now that $\mathbb{P}[0, 1 \in \pi_0] > 0$, the same argument shows that $s_0 > 0$. We conclude by exchangeability. The assertion *(iv)* is straigthforward : if some blocks not containing 0 are singletons then $\mathbb{P}(U_1 \in [0,1]\backslash\mathcal{V}) > 0$ and so $1 - \sum_{i=0}^{\infty} s_i > 0$. The assertion *v)* is trivial.□

It remains to see whether the distinguished paint-box construction of Definition 1.3 yields all the exchangeable distinguished partitions.

Theorem 1.5 *The following assertions are equivalent :*

i) π is a random distinguished exchangeable partition

ii) There exists a random distinguished mass-partition $S = (S_0, S_1, ...)$ such that conditionally given $S = \boldsymbol{s}$, π has the law of an $\boldsymbol{s}$-distinguished paint box $(\rho_{\boldsymbol{s}})$. Further $|\pi|^{\downarrow} = S$.

Proof. A mixture of distinguished paint-boxes is still exchangeable, this shows that *ii)* implies *i)*. Let π be exchangeable. We adapt a proof of Aldous [Ald85], see also [Ber06] p101. We call selection map, any random function $b : \mathbb{Z}_+ \to \mathbb{Z}_+$ that maps all the points of the block π_0 to 0, and all the points of a block π_i for $i \geq 1$ to the same point of that block. Let $U_0 = 0$, $(U_i)_{i\geq 1}$ be i.i.d uniform on $[0,1]$, independent of Π and of the selection map b. We define $X_n = U_{b(n)}$. The law of $(X_n, n \geq 1)$ does not depend on the choice of b. The key of the proof is the exchangeability of $(X_n)_{n\geq 1}$. Let σ be a permutation with $\sigma(0) = 0$, we have

$$X_{\sigma(n)} = U_{b(\sigma(n))} = U'_{b'(n)},$$

where $U'_i = U_{\sigma(i)}$ and $b' = \sigma^{-1} \circ b \circ \sigma$. We verify that b' is a selection map for the partition $\sigma\pi$. Let $i \geq 0$, and $n \in \sigma\pi_i$, by definition of $\sigma\pi$, $\sigma\pi_i = \sigma^{-1}(\pi_{\eta(i)})$ for a certain permutation η such that $\eta(0) = 0$, then there exists $k \in \pi_{\eta(i)}$ such that $n = \sigma^{-1}(k)$. For $i = 0$, $k \in \pi_0$ and then $b'(n) = \sigma^{-1}(b(k)) = \sigma^{-1}(0) = 0$ for all $n \in \sigma\pi_0$. For $i \geq 1$, we clearly have that $b'(n) = \sigma^{-1} \circ b(k)$ depends only on i.
The sequence $(U'_i, i \geq 1)$ has the same law as $(U_i, i \geq 1)$. By exchangeability and independence of (U_j) and π : $((U'_n)_{n\geq 1}, \sigma\pi)$ has the same law as $((U_n)_{n\geq 1}, \pi)$, and the sequence $(X_n, n \geq 1)$ is exchangeable. By the de Finetti theorem, we have that conditionally on the random probability measure $\rho := \lim_{n\to\infty} \frac{1}{n} \sum_{i=1}^{n} \delta_{X_i}$, $(X_n, n \geq 1)$ are i.i.d random variables with distribution ρ. Moreover, by the definition of X_n, $i \underset{\pi}{\sim} j$ if and only if $X_i = X_j$. We deduce that conditionally given $\rho = \alpha$, the distinguished partition π is a $s(\alpha)$-distinguished paint box with $s(\alpha) := (\alpha(0), \alpha(x_1), ...)$. By the distinguished paint-box construction, on $\{\rho = \alpha\}$, $s(\alpha)$ is the mass-partition of π and so $|\pi|^{\downarrow} = s(\rho)$, moreover the random sequence $S := s(\rho)$ in $\mathcal{P}^0_{\mathbf{m}}$ verifies the assertion *ii)*. To conclude the random partition π has the law of a ν-mixture of distinguished paint-boxes, where ν is the law of $s(\rho) = |\pi|^{\downarrow}$. □

This theorem sets up a bijection between probability distributions for exchangeable distinguished partitions and probability distributions on the space of distinguished mass-partitions, $\mathcal{P}^0_{\mathbf{m}}$:

$$\mathbb{P}[\pi \in .] = \int_{\mathcal{P}^0_m} \rho_{\mathbf{s}}(.)\mathbb{P}(|\pi|^{\downarrow} \in ds).$$

Remark 1 *Let π be an exchangeable distinguished partition, as for exchangeable partitions, see [Pit06], we can show that there exists a function p such that*

$$\mathbb{P}[\pi_{|[n]} = (B_0, ..., B_k)] = p(n_0, ..., n_k).$$

where $n_i = \#B_i$, $i = 0, ..., k$. *Contrary to exchangeable random partitions, the function* p *is not totally symmetric but only invariant by permutations of the arguments* $(n_1, ..., n_k)$. *Indeed by exchangeability* $\mathbb{P}[\pi = (\sigma B_0, ..., \sigma B_k)] = p(\#B_0, ..., \#B_k) = p(\#B_{\eta(0)}, ..., \#B_{\eta(k)})$ *where* η *is the permutation such that* $\sigma\pi_i = \sigma^{-1}(\pi_{\eta(i)})$. *Due to the assumption* $\sigma(0) = 0$, *the permutation* η *is such that* $\eta(0) = 0$. *The exchangeable distinguished partitions are thus special cases of partially exchangeable partitions in the sense of [Pit95].*
We mention that Donnelly and Joyce, [DJ91] defined the exchangeable ordered partitions for which all the blocks are distinguished (they speak about exchangeable random ranking). They obtain a Kingman's representation for the exchangeable ordered partition structure. For every μ *probability on* $[0, 1]$, *an "ordered paint-box" is constructed from a sequence of i.i.d.* μ *variables. We stress that, contrary to an exchangeable distinguished paint-box, the law of a* μ*-ordered paint-box depends on* ***the order of the atoms*** *of* μ. *Exchangeable ordered partitions are also partially exchangeable but the function* p *has no symmetry properties. Gnedin in [Gne97] gives a representation of exchangeable compositions which are a generalization of exchangeable random rankings.*
We could define distinguished partitions with several distinguished blocks. It corresponds to a population with several sources of immigration : each distinguished block gathers the progeny of an immigration source. For sake of simplicity, we distinguish here just one block.

In the next section, we define distinguished coalescents which can be interpreted as a genealogy for a population with immigration.

1.3 Distinguished coalescents

Imagine an infinite haploid population with immigration. We denote by $\Pi(t)$ the partition of the current population into families having the same ancestor t generations earlier. As explained in the Introduction, individuals who have no ancestor in the population at generation t, form the distinguished block of $\Pi(t)$. Actually, the individual 0 can be viewed as their common ancestor. When some individuals have the same ancestor at a generation t, they have the same ancestor at any generation $t' \geq t$. In terms of partitions, all integers in the same block of $\Pi(t)$, are in the same block of $\Pi(t')$ for any $t' \geq t$. The collection of partitions $(\Pi(t))_{t \geq 0}$ will be a coalescent process. To define these processes and go from an exchangeable distinguished partition to another coarser partition, we have to introduce the coagulation operator.

1.3.1 Coagulation operator and distinguished coalescents

To define the distinguished exchangeable coalescents, we need to define an operator on the space of distinguished partitions.

Definition 1.6 *Let* $\pi, \pi' \in \mathcal{P}^0_\infty$, *the partition* $coag(\pi, \pi')$ *is defined by* $coag(\pi, \pi')_i = \pi''_i$ *where* $\pi''_i = \bigcup_{j \in \pi'_i} \pi_j$. *The partition* $coag(\pi, \pi')$ *is exactly the one obtained by coagulating blocks of* π *according to blocks of* π'.

We denote by $0_{[\infty]}$ the partition into singletons $\{\{0\}, \{1\}, ...\}$, and by $1_{[\infty]}$ the trivial partition $\{\mathbb{Z}^+, \emptyset, ...\}$. Plainly, for all $n \geq 0$, $coag(\pi, \pi')_{|[n]} = coag(\pi_{|[n]}, \pi'_{|[n]})$ and for all $\pi \in \mathcal{P}^0_\infty, coag(\pi, 0_{[\infty]}) = \pi$ and $coag(\pi, 1_{[\infty]}) = 1_{[\infty]}$. Note that however, we do not have $coag(\pi, \pi')_{|K} = coag(\pi_{|K}, \pi'_{|K})$ for $K \subset \mathbb{N}$ in general.

Proposition 1.7 *Let π, π' be two independent exchangeable distinguished partitions. The distinguished partition coag(π, π') is also exchangeable.*

Proof. See the proof of Lemma 4.3 in [Ber06]. An alternative proof is provided in Chapter 3, Remark 5.
The coagulation operator allows us to define distinguished coalescents which are Markovian processes valued in distinguished partitions of $\mathbb{N}$.

Definition 1.8 *A Markov process Π with values in $\mathcal{P}^0_\infty$ is called a distinguished coalescent if its semi-group is given as follows : the conditional law of $\Pi(t+t')$ given $\Pi(t) = \pi$ is the law of coag(π, π') where π' is some exchangeable distinguished partition (whose law only depends on t'). A distinguished coalescent is called standard if $\Pi(0) = 0_{[\infty]}$.*

Plainly, the random partition $\Pi_{|\mathbb{N}}(t)$ is exchangeable for all $t \geq 0$. However, we stress that in general the process $(\Pi_{|\mathbb{N}}(t), t \geq 0)$ is not an exchangeable coalescent and not even Markovian. We will give an example in Section 4.2.

For every $n \geq 1$, the restriction $(\Pi_{|[n]}(t), t \geq 0)$ is a continuous time Markov chain with a semi-group given by the operator *coag*. Let $\pi \in \mathcal{P}^0_n \setminus \{0_{[n]}\}$, we denote by q_π, the jump rate of $\Pi_{|[n]}$ from $0_{[n]}$ to π :

$$q_\pi := \lim_{t \to 0+} \frac{1}{t} \mathbb{P}_{0_{[n]}}[\Pi_{|[n]}(t) = \pi].$$

Definition 1.9 *The distinguished coagulation measure of Π is the unique measure μ on $\mathcal{P}^0_\infty$, such that $\mu(\{0_{[\infty]}\}) = 0$ and*

$$\mu(\mathcal{P}^0_{\infty,\pi}) = q_\pi$$

for every $n \in \mathbb{Z}_+$ and every partition $\pi \in \mathcal{P}^0_n$.
Moreover, the measure μ fulfills

$$\mu(\pi \in \mathcal{P}^0_\infty : \pi_{|[n]} \neq 0_{[n]}) < \infty \text{ and } \mu \text{ is exchangeable.}$$

Conversely, any measure fulfilling the previous conditions will be called a distinguished coagulation measure.

Proof : existence and uniqueness. An easy adaptation of the proof of Proposition 4.4 in [Ber06] gives the existence and uniqueness of the distinguished coagulation measure. The laws of $(\Pi_{|[n]}(t), t \geq 0)$, for $n \geq 1$ are compatible, and then the family of jump rates of $\Pi_{|[n]}$ characterizes the law of $(\Pi(t), t \geq 0)$. Using the compatibility property, we get for all $\pi \in \mathcal{P}^0_n$, and $n' \geq n$:

$$q_\pi = \sum_{\pi' \in \mathcal{P}^0_{n',\pi}} q_{\pi'}.$$

Moreover,

$$\mathcal{P}^0_{\infty,\pi} = \bigcup_{\pi' \in \mathcal{P}^0_{n',\pi}} \mathcal{P}^0_{\infty,\pi'},$$

where the union is over disjoint subsets, the function μ is then additive. The class of finite unions of sets $\mathcal{P}^0_{\infty,\pi}$ is a ring and generates the Borelian σ-field. Indeed, we have

$$\mathcal{P}^0_{\infty,\pi} \setminus \mathcal{P}^0_{\infty,\pi'} = \bigcup_{\pi'' \in \mathcal{P}^0_{n'',\pi},\ \pi''_{|[n']} \neq \pi'} \mathcal{P}^0_{\infty,\pi''}.$$

We conclude by an application of Caratheodory's extension theorem that this defines a unique measure μ on $\mathcal{P}^0_\infty$. (see proposition 3.2 of [Ber06]). Moreover, the measure μ fulfills the following properties :

i) μ is invariant by permutations which leave 0 unchanged

ii) $\mu(\{0_{[\infty]}) = 0$ and $\forall n \geq 0$, $\mu(\{\pi \in \mathcal{P}^0_\infty; \pi_{|[n]} \neq 0_{[n]}\}) < \infty$.

The first assertion *i)* is plain. For *ii)*, the total jump rate of Π restricted to $[n]$ is exactly $\mu(\{\pi \in \mathcal{P}^0_\infty; \pi_{|[n]} \neq 0_{[n]}\})$. This quantity is finite because $\Pi_{|[n]}$ is a $\mathcal{P}^0_n$-valued Markov chain. □

Let μ be a distinguished coagulation measure, we construct explicitly a distinguished coalescent process with coagulation measure (in the sense of Definition 1.9) μ. Let N be a Poisson measure with intensity $dt \otimes \mu(d\pi)$. Let N_b be the image of N by the map $(t, \pi) \mapsto (t, \pi_{|[b]})$. Its intensity, denoted by μ_b, is the image of μ by the previous map. We denote by (t_i, π_i) the atoms of N_b and define a process $(\Pi^b(t), t \geq 0)$ by the following recursion :

– For all $0 \leq t < t_1$ $\Pi^b(t) = 0_{[b]}$,

– if $t_i \leq t < t_{i+1}$, $\Pi^b(t) = coag(\Pi^b(t_{i-1}), \pi^i)$, with $t_0 = 0$.

Proposition 1.10 *The sequence of random partitions* $(\Pi^b(t), b \in \mathbb{N})$ *is compatible, which means that for all* $a \leq b$, $\Pi^b_{|[a]} = \Pi^a$. *The unique process* $(\Pi(t), t \geq 0)$ *such that* $\Pi_{|[b]}(t) = \Pi^b(t)$, *defined by* $\Pi_i(t) = \bigcup_{b \geq 1} \Pi^b_i(t)$ *is a distinguished coalescent with coagulation measure* μ.

Proof. Same arguments as those of Proposition 4.5 in [Ber06] apply. We want to show that $\Pi^{b-1}(t) = \Pi^b_{|[b-1]}(t)$ for all $b \geq 1$. Let (t_1, π^1) be the first atom of N_b on $[0, \infty[\times \mathcal{P}^0_b \backslash \{0_{[b]}\}$. Plainly, all atoms of N_{b-1} are atoms of N_b, and so there is no atom of N_{b-1} on $[0, t_1[$: $\Pi^{b-1}(t) = \Pi^b_{|[b-1]}(t)$, $\forall t \in [0, t_1[$. Some atoms of N_b are atoms of N_{b-1} and then we have to consider the two following cases : firstly, if $\pi^{(1)}_{|[b-1]} \neq 0_{[b-1]}$ then $(t_1, \pi^1_{|[b-1]})$ is the first atom of N_{b-1} on $[0, \infty[\times \mathcal{P}^0_b \setminus \{0_{[b-1]}\}$. We have, by Definition 1.6, $\Pi^b_{|[b-1]}(t) = coag(\Pi^b(0), \pi^1)_{|[b-1]} = \Pi^{b-1}(t)$ $\forall t \in [t_1, t_2[$. Secondly, if $\pi^1_{|[b-1]} = 0_{[b-1]}$ then (t_1, π^1) is not an atom of N_{b-1}, $\Pi^{b-1}(t) = 0_{[b-1]} = \Pi^b_{|[b-1]}(t)$, $\forall t \in [0, t_2[$. Recursively, for all $t_i \leq t < t_{i+1}$

$$\Pi^b_{|[b-1]}(t) = coag(\Pi^b(t_{i-1}), \pi^i)_{|[b-1]} = coag(\Pi^b_{|[b-1]}(t_{i-1}), \pi^i_{|[b-1]})$$

And so, we repeat the previous discussion about $\pi^i_{|[b-1]}$. To conclude, we claim that by construction, the semi-group of $(\Pi(t), t \geq 0)$ verifies the condition of the definition. □

Example 1 *We denote by* $K(i, j)$ *the simple distinguished partition where* i *and* j *are in the same block and all the other blocks are singletons. Let* c_0, c_1 *be two non-negative real numbers. The measure* $\mu = c_0\mu^K_0 + c_1\mu^K_1$, *where* $\mu^K_0 := \sum_{1 \leq i} \delta_{K(0,i)}$ *and* $\mu^K_1 := \sum_{1 \leq i < j} \delta_{K(i,j)}$, *is a distinguished coagulation measure. The process obtained is called the Kingman's distinguished coalescent with rates* (c_0, c_1).

Indeed, the measure μ defined as above is plainly a distinguished coagulation measure. The Poissonian construction explains the dynamics of this process. At a constant rate c_0, a block not containing 0 merges with Π_0 that is a singular coagulation with the distinguished block. At a constant rate c_1, two blocks not containing 0 merge into one, that is the classic binary coagulation of Kingman's coalescent.

Proposition 1.11 *The distinguished coalescents are Feller processes, which means that the map* $\pi \in \mathcal{P}_\infty^0 \mapsto \mathbb{E}[\phi(coag(\pi, \Pi(t))]$ *is continuous and that*

$$\lim_{t\to 0} \mathbb{E}[\phi(coag(\pi, \Pi(t))] = \phi(\pi).$$

Proof. Recall the distance d defined in section 2. If $d(\pi, \pi') \le \epsilon$ then

$$d(coag(\pi, \Pi(t)), coag(\pi', \Pi(t))) \le \epsilon.$$

By continuity of ϕ, and Lebesgue's theorem, $\pi \mapsto \mathbb{E}[\phi(coag(\pi, \Pi(t))]$ is continuous. Moreover, $\Pi(0) = 0_{\mathbb{Z}_+}$.□

Therefore the process has a càdlàg version and is strong Markovian. The completed natural filtration of Π denoted by $(\mathcal{F}_t)_{t\ge 0}$ is right continuous.

1.3.2 Characterization in law of the distinguished coalescents

The next theorem is one of the main results of this work, it claims that the law of a distinguished coalescent is characterized by two non-negative real numbers c_0, c_1 and a measure ν on $\mathcal{P}_{\mathbf{m}}^0$. It should be viewed as an extension of Theorem 1.5 to certain infinite measures. Recall that $\rho_{\mathbf{s}}$ denotes the law of an exchangeable distinguished $\mathbf{s}$-paint-box for $\mathbf{s} \in \mathcal{P}_{\mathbf{m}}^0$.

Theorem 1.12 *Recalling Definition 1.9, let μ be a distinguished coagulation measure. There exist two unique real numbers c_0, c_1 and a unique measure ν on $\mathcal{P}_{\boldsymbol{m}}^0$ which fulfills :*

$$\nu(0) = 0 \text{ and } \int_{\mathcal{P}_{\boldsymbol{m}}^0} (s_0 + \textstyle\sum_{i=1}^\infty s_i^2)\nu(ds) < \infty$$

such that

$$\mu = c_0\mu_0^K + c_1\mu_1^K + \rho_\nu$$

where

$$\rho_\nu(.) := \int_{s\in\mathcal{P}_{\boldsymbol{m}}^0} \rho_{\boldsymbol{s}}(.)\nu(ds).$$

Conversely, let c_0, c_1 and ν be two real numbers and a measure on $\mathcal{P}_{\boldsymbol{m}}^0$ verifying the previous conditions, there exists a unique (in law) distinguished coalescent with $\mu = c_0\mu_0^K + c_1\mu_1^K + \rho_\nu$.

When ν is carried on $\{s \in \mathcal{P}_{\mathbf{m}}^0; s_0 = 0\}$ (which can be identified as $\mathcal{P}_{\mathbf{m}}$), the block containing 0 is reduced to the singleton $\{0\}$ (we distinguish no block) and considering the restriction to $\mathbb{N}$, we recover the characterization of exchangeable coalescents (also called Ξ-coalescents) by Schweinsberg in [Sch00a].

Proof. Arguments used to prove this theorem are adapted from those of Theorem 4.2 in Chapter 4 of [Ber06]. Nevertheless, we give details to highlight the fact that the condition on ν differs from that of Theorem 4.2 in [Ber06]. We denote by μ_n the restriction of μ

to $\{\pi \in \mathcal{P}^0_\infty; \pi_{|[n]} \neq 0_{[n]}\}$. The measure $\mu_n = 1_{\{\pi\in\mathcal{P}^0_\infty;\pi_{|[n]}\neq 0_{[n]}\}}\mu$ has a finite mass and is invariant under the action of permutations σ that coincide with the identity on $[n]$. We define the n-shift on distinguished partitions by the map $\pi \to \pi'$ defined by :

$$\forall i,j \geq 1 : \quad i \underset{\pi'}{\sim} j \Longleftrightarrow i+n \underset{\pi}{\sim} j+n$$
$$\forall j \geq 1 : \quad 0 \underset{\pi'}{\sim} j \Longleftrightarrow 0 \underset{\pi}{\sim} j+n.$$

The image of μ_n by the n-shift, denoted by $\bar{\mu}_n$, is invariant under the action of permutations σ of $\mathbb{Z}_+$ such that $\sigma(0)=0$. By the Kingman's correspondence (Theorem 1.5)

$$\bar{\mu}_n(d\pi) = \int_{\mathcal{P}^0_{\mathbf{m}}} \rho_{\mathbf{s}}(d\pi)\bar{\mu}_n(|\pi|^{\downarrow} \in ds).$$

Moreover, $\bar{\mu}_n$ almost every partition has asymptotic frequencies. The shift does not affect asymptotic frequencies and then μ_n a.e partition has asymptotic frequencies. The measure μ is the increasing limit of the μ_n, we deduce that μ almost every partition has asymptotic frequencies.
By the distinguished paint-box representation of $\bar{\mu}_n$, we get for all $\mathbf{s} \in \mathcal{P}^0_{\mathbf{m}} \setminus \{0\}$

$$\mu_n(n+1 \sim n+2 \not\sim 0 \text{ or } 0 \sim n+1)||\pi|^{\downarrow} = s) = s_0 + \textstyle\sum_{k=1}^{\infty} s_k^2.$$

Let ν_n be the image of μ_n by the map $\pi \mapsto |\pi|^{\downarrow}$

$$\nu_n(ds) = \mu_n(|\pi|^{\downarrow} \in ds).$$

We stress that $\nu_n(ds) = \bar{\mu}_n(|\pi|^{\downarrow} \in ds)$ because the n-shift has no impact on asymptotic frequencies.
On the one hand, we have :

$$\mu_n(n+1 \sim n+2 \not\sim 0 \text{ or } 0 \sim n+1) \geq \int_{\mathcal{P}^0_{\mathbf{m}}} (s_0 + \sum_{i=1}^{\infty} s_i^2)\nu_n(ds).$$

On the other hand,

$$\mu_n(n+1 \sim n+2 \not\sim 0 \text{ or } 0 \sim n+1) \leq \mu(n+1 \sim n+2 \not\sim 0 \text{ or } 0 \sim n+1).$$

By exchangeability of μ

$$\mu(n+1 \sim n+2 \not\sim 0 \text{ or } 0 \sim n+1) = \mu(1 \sim 2 \not\sim 0 \text{ or } 0 \sim 1) \text{ and}$$

$$\mu(1 \sim 2 \not\sim 0 \text{ or } 0 \sim 1) \leq \mu(\pi_{|[2]} \neq 0_{[2]}) < \infty.$$

We deduce that the finite measures ν_n increase as $n \uparrow \infty$ to the measure $\nu := \mu(|\pi|^{\downarrow} \in ds)$ and so

$$\lim_{n\to\infty} \int_{\mathcal{P}^0_{\mathbf{m}}} (s_0 + \sum_{i=1}^{\infty} s_i^2)\nu_n(ds) = \int_{\mathcal{P}^0_{\mathbf{m}}} (s_0 + \sum_{i=1}^{\infty} s_i^2)\nu(ds) \leq \mu(\pi_{|[2]} \neq 0_{[2]}) < \infty.$$

Let $k \in \mathbb{N}$ and $\pi^{[k]} \in \mathcal{P}^0_k \setminus \{0_{[k]}\}$. The sequence of events $\pi_{|\{k+1,\ldots,k+n\}} \neq 0_{\{k+1,\ldots,k+n\}}$ is increasing with n, then we have

$$\mu(\pi_{|[k]} = \pi^{[k]}, |\pi|^{\downarrow} \neq 0) = \lim_{n\to\infty} \mu(\pi_{|[k]} = \pi^{[k]}, |\pi|^{\downarrow} \neq 0, \pi_{|\{k+1,\ldots,k+n\}} \neq 0_{\{k+1,\ldots,k+n\}}).$$

By an obvious permutation we get

$$\mu(\pi_{|[k]} = \pi^{[k]}, |\pi|^{\downarrow} \neq 0, \pi_{|\{k+1,\ldots,k+n\}} \neq 0_{\{k+1,\ldots,k+n\}}) = \bar{\mu}_n(\pi_{|[k]} = \pi^{[k]}, |\pi|^{\downarrow} \neq 0).$$

Thus, using the distinguished paint-box representation of $\bar{\mu}_n$, we deduce that

$$\bar{\mu}_n(\pi_{|[k]} = \pi^{[k]}, |\pi|^{\downarrow} \neq 0) = \int_{\mathcal{P}^0_{\mathbf{m}}} \rho_{\mathbf{s}}(\pi_{|[k]} = \pi^{[k]}, |\pi|^{\downarrow} \neq 0)\nu_n(ds)$$
$$\underset{n\to\infty}{\longrightarrow} \int_{\mathcal{P}^0_{\mathbf{m}}} \rho_{\mathbf{s}}(\pi_{|[k]} = \pi^{[k]}, |\pi|^{\downarrow} \neq 0)\nu(ds).$$

As k is arbitrary, we get

$$1_{\{|\pi|^{\downarrow}\neq 0\}}\mu(d\pi) = \int_{\mathcal{P}^0_{\mathbf{m}}} \rho_{\mathbf{s}}(d\pi)\nu(ds).$$

It remains to study $1_{\{|\pi|^{\downarrow}=0\}}\mu(d\pi)$. Consider now $\tilde{\mu}(d\pi) := 1_{\{0\sim 1, |\pi|^{\downarrow}=0\}}\mu(d\pi)$ which has finite mass (because $\mu(0 \sim 1) < \infty$). We want to show that $\tilde{\mu}(d\pi)$ is proportional to $\delta_{K(0,1)}$, where $K(0,1)$ is the simple partition with $0 \sim 1$. Let $\tilde{\mu}_2(d\pi)$ be the image of $\tilde{\mu}$ by the 2-shift. The measure $\tilde{\mu}_2(d\pi)$ is supported by $\{\pi \in \mathcal{P}_\infty; |\pi|^{\downarrow} = 0\}$ and is exchangeable with finite mass. By the distinguished paint-box construction, the only exchangeable partition with asymptotic frequencies $|\pi|^{\downarrow} = 0$ is the partition into singletons $0_{[\infty]}$. Therefore, $\tilde{\mu}_2(d\pi) = c_0\delta_{\{\{0\},\{1\},\ldots\}}$. We deduce that for $\tilde{\mu}_2$-almost every π', $\forall i \neq j$, $i \underset{\pi'}{\not\sim} j$. From the definition of the 2-shift, we get that for $\tilde{\mu}$-almost every π

$$\forall i, j \geq 1\,;\, i \neq j,\ i+2 \underset{\pi}{\not\sim} j+2, \text{ and } \forall j \geq 1,\ 0 \underset{\pi}{\not\sim} j+2.$$

It implies that we have to consider only three possibilities

$$\tilde{\mu}(d\pi) = c_0\delta_{K(0,1)} \text{ or } \tilde{\mu}(0 \sim 1 \sim 2) > 0 \text{ or } \tilde{\mu}(2 \sim k) > 0 \text{ for some } k \geq 3.$$

If $\tilde{\mu}(2 \sim k) > 0$ for some $k \geq 3$, we get by exchangeability that $\tilde{\mu}(2 \sim k) = \tilde{\mu}(2 \sim 3) > 0$. Moreover, the collection of sets $\{\{2 \sim n\}, n \geq 3\}$ is such that the intersection of two or more sets has a zero measure $\tilde{\mu}$ and then $\tilde{\mu}(\cup_{n\geq 3}\{2 \sim n\}) = \sum_{n\geq 3} \tilde{\mu}(2 \sim n) \leq \mu(0 \sim 1)$. It follows that $\mu(0 \sim 1) = \infty$. This is a contradiction because $\mu(0 \sim 1) < \infty$.

If $\tilde{\mu}(0 \sim 1 \sim 2) > 0$, then by exchangeability for all $n \geq 2$, $\tilde{\mu}(0 \sim 1 \sim n) = c_0 > 0$ and by the same arguments, the same contradiction appears. We deduce that $\tilde{\mu}(d\pi)$ is equal to $c_0\delta_{K(0,1)}$. By exchangeability, we have $1_{\{0\sim i, |\pi|^{\downarrow}=0\}}\mu(d\pi) = c_0\delta_{K(0,i)}$. The measure $\mu 1_{\{|\pi|^{\downarrow}=0,\ \pi_0\neq\{0\}\}}$ is carried on simple partition π such that π_0 is not singleton, moreover the collection of sets $\{\{0 \sim i\}, i \geq 1\}$ is such that the intersection of two sets has a zero measure. Therefore, we have

$$1_{\{|\pi|^{\downarrow}=0,\ \pi_0\neq\{0\}\}}\mu(d\pi) = c_0 \sum_{i\geq 1} \delta_{K(0,i)}.$$

The restriction of μ to $\{\pi \in \mathcal{P}^0_\infty; \pi_0 = \{0\}\}$ can be viewed as an exchangeable measure on $\mathcal{P}_\infty$, the argument to conclude is then the same as in [Ber06] on page 184. Namely following the same reasoning as above we may show

$$1_{\{1\sim 2, \pi_0=\{0\}; |\pi|^{\downarrow}=0\}}\mu(d\pi) = c_1\delta_{K(1,2)}$$

for some $c_1 \geq 0$ and by exchangeability, we get

$$1_{\{\pi_0=\{0\}; |\pi|^{\downarrow}=0\}}\mu(d\pi) = c_1 \sum_{i<j} \delta_{K(i,j)}.$$

□

Proposition 1.13 *Denoting by $(D(t), t \geq 0) := (1 - \sum_{i=0}^{\infty} |\Pi_i(t)|)_{t\geq 0}$ the process of dust of Π. The arguments of [Pit99] or [Ber06] allow us to show that for all $t > 0$ the random partition $\Pi(t)$ has improper asymptotic frequencies with a strictly positive probability if and only if $c_1 = 0$ and $\int_{\mathcal{P}_m^0} (\sum_{i=1}^{\infty} s_i)\, \nu(ds) < \infty$.*

In that case, the process $(\xi(t), t \geq 0) := (-ln(D(t)), t \geq 0)$ is a subordinator with Laplace exponent

$$\phi(q) = c_0 q + \int_{\mathcal{P}_m^0} (1 - \delta^q)\nu(ds).$$

Note that the drift coefficient c_0 may be positive, which contrasts with the result of Pitman [Pit99].

Proof. The random partition $\Pi(t)$ has proper asymptotic frequencies a.s if and only if almost surely, there are no singleton blocks not containing 0. By exchangeability, this is equivalent to $\mathbb{P}[\Pi_1(t) = \{1\}] = 0$. Let N be a Poisson Point Process on $\mathcal{P}_\infty^0$ with intensity μ. By the Poissonian construction, the event $\{\Pi_1(t) = \{1\}\}$ occurs if and only if all the atoms of N on $[0, t]$ fulfill $\pi_1 = \{1\}$; we get

$$\mathbb{P}[\Pi_1(t) = \{1\}] = \mathbb{P}\left[N([0, t] \times \{\pi \in \mathcal{P}_\infty; \pi_1 \neq \{1\}\}) = 0\right] = \exp(-ta)$$

where

$$a = \mu(\{\pi; \pi_1 \neq \{1\}\}).$$

By Theorem 1.12, it follows that

$$a = \infty 1_{c_1>0} + c_0 + \rho_\nu(\pi_1 \neq \{1\})$$

By exchangeability,

$$\rho_\nu(\pi_1 \neq \{1\}) = \textstyle\int_{\mathcal{P}_\mathbf{m}^0} \mathbb{P}\left(U_1 \text{ does not fall in the dust }\right)\nu(ds) = \int_{\mathcal{P}_\mathbf{m}^0} (1 - \delta)\nu(ds).$$

We then have

$$a < \infty \text{ if and only if } c_1 = 0 \text{ and } \textstyle\int_{\mathcal{P}_\mathbf{m}^0} (1 - \delta)\nu(ds) < \infty.$$

The proof of the stationarity and independence of multiplicative increments of $(D(t), t \geq 0)$ is as in Chapter 4 of [Ber06] (page 189). We just give the key points. First we may observe that if π and π' are two distinguished partitions which possess asymptotic frequencies, such that the blocks with zero asymptotic frequency are either empty or singletons, then

$$|coag(\pi, \pi')_i| = \left(1 - \sum_{j=0}^{\infty} |\pi_j|\right) |\pi'_i| + \sum_{j\in\pi'_i} |\pi_j|.$$

We have :

$$\begin{aligned}
D(t+s) &= 1-\sum_{i=0}^{\infty}|\Pi_i(t+s)| = 1-\sum_{i=0}^{\infty}|coag(\Pi(t),\tilde{\Pi}(s))_i| \\
&= 1-\sum_{i=0}^{\infty}\left[\left(1-\sum_{j=0}^{\infty}|\Pi_j(t)|\right)|\tilde{\Pi}_i(s)| + \sum_{j\in\tilde{\Pi}_i(s)}|\Pi_j(t)|\right] \\
&= 1-D(t)\sum_{i=0}^{\infty}|\tilde{\Pi}_i(s)| - \sum_{i=0}^{\infty}\sum_{j\in\tilde{\Pi}_i(s)}|\Pi_j(t)| \\
&= 1-D(t)\sum_{i=0}^{\infty}|\tilde{\Pi}_i(s)| - \sum_{i=0}^{\infty}|\Pi_i(t)| \\
&= D(t)\tilde{D}(s).
\end{aligned}$$

We focus now on the Laplace exponent of ξ. Let $q \geq 1$, $\mathbf{s} \in \mathcal{P}^0_{\mathbf{m}}$, thanks to the paint-box structure, $\rho_{\mathbf{s}}\,(1,...,q$ are singletons $) = \delta^q$, and then $\rho_{\mathbf{s}}(\{\pi; \exists 1 \leq i \leq q$ not singleton of $\pi\}) = 1-\delta^q$. Thus, we compute all of the moments of $D(t)$

$$\mathbb{E}[D(t)^q] = \mathbb{P}[1,...,q \text{ are singletons of } \Pi(t)].$$

By exchangeability and Poissonian structure, the probability of the event

$$\{1,...,q\} \text{ are singletons of } \Pi(t)$$

is the probability that $N([0,t]\times\{\exists 1 \leq i \leq q \text{ not singleton of } \pi\}] = 0$. By Poissonian calculations, we get

$$\mathbb{E}[D(t)^q] = \exp\left(-t\mu(\{\pi; \exists 1 \leq i \leq q \text{ which is not a singleton of } \pi\})\right).$$

By Theorem 1.12

$$\mu(\{\pi; \exists 1 \leq i \leq q \text{ not singleton of } \pi\}) = c_0 q + \textstyle\int_{\mathcal{P}^0_{\mathbf{m}}}(1-\delta^q)\nu(ds)$$

and then

$$\mathbb{E}[D(t)^q] = \exp\left[-t\left(c_0 q + \int_{\mathcal{P}^0_{\mathbf{m}}}(1-\delta^q)\nu(ds)\right)\right].$$

We thus have $\mathbb{E}[D(t)^q] = \mathbb{E}[e^{-q\xi(t)}] = e^{-t\phi(q)}$ with ϕ as in the statement. $\square$

Remark 2 *The exchangeability implies that a distinguished coalescent* $(\Pi(t), t \geq 0)$ *may have proper asymptotic frequencies with the distinguished block* $\Pi_0(t)$ *singleton for some* $t \geq 0$. *We compute the probability that this event occurs* $\mathbb{P}[\Pi_0(t) = \{0\}] = \mathbb{P}[N([0,t]\times\{\pi \in \mathcal{P}_\infty; \pi_0 \neq \{0\}\} = 0] = \exp(-tb)$, *where*

$$b = \mu(\{\pi; \pi_0 \neq \{0\}\}) = \infty 1_{c_0>0} + \nu(s \in \mathcal{P}^0_{m}; s_0 > 0).$$

We deduce that if $c_0 = 0$ *and* $\nu(s \in \mathcal{P}^0_{m}; s_0 > 0) < \infty$, *the distinguished block stays singleton for a strictly positive time.*

1.4 The simple distinguished exchangeable coalescents : M-coalescents

In this section, we focus on simple distinguished coalescent for which the coagulation measure μ is carried by the set of simple distinguished partitions. We call them, hereafter, M-coalescents. These processes are the analogue of Λ-coalescents for exchangeable distinguished coalescents. Historically, the Λ-coalescent is the first exchangeable coalescent with multiple collisions to have been defined, see Pitman, [Pit99] and Sagitov, [Sag99]. We begin by recalling some basic facts about Λ-coalescents.

1.4.1 Λ-coalescents

A Λ-coalescent (also called simple exchangeable coalescent) is a process taking values in the partitions of $\mathbb{N}$ describing the genealogy of an infinite haploid population, labelled by $\mathbb{N}$ where two or more ancestral lineages merging cannot occur simultaneously. We stress that in these coalescent processes, each individual has an ancestor in the population. Immigration phenomenon is not taken into account and no block is distinguished. A simple exchangeable coalescent is a Markovian process $(\Pi(t), t \geq 0)$ on the space of partitions of $\mathbb{N}$ satisfying :

i) If $n \in \mathbb{N}$, then the restriction $(\Pi_{|[n]}(t))$ is a continuous-time Markov chain valued in $\mathcal{P}_n$;

ii) For each n, $(\Pi_{|[n]}(t))$ evolves by exchangeable merging of blocks :

$$\Pi_{|[n]}(t) = coag(\Pi_{|[n]}(t-), \pi')$$

where π' is an independent simple exchangeable partition ;

By Theorem 1 in Pitman [Pit99], or Sagitov [Sag99], we know that any simple exchangeable coalescent is characterized in law by a finite measure Λ on $[0, 1]$. The dynamics of Π can be described as follows : whenever $\Pi_{|[n]}(t)$ is a partition with b blocks, the rate at which a k-tuple of its blocks merges is

$$\lambda_{b,k} = \int_0^1 x^{k-2}(1-x)^{b-k}\Lambda(dx).$$

When Λ is the Dirac at 0, we recover the Kingman's coalescent. When $\Lambda(\{0\}) = 0$, the Λ-coalescent can be constructed via a Poisson Point Process on $\mathbb{R}_+ \times [0, 1]$ with intensity $dt \otimes \nu(dx)$, where $\nu(dx) = x^{-2}\Lambda(dx)$:

$$N = \sum_{i \in \mathbb{N}} \delta_{(t_i, x_i)}.$$

The atoms of N encode the evolution of the coalescent Π : At time $t-$, flip a coin with probability of "heads" x for each block. All blocks flipping "heads" are merged immediately. We can, also, construct a simple exchangeable partition, π' where the non trivial block is constituted by indices of "heads". Thus, we get $\Pi(t)$ by $coag(\Pi(t-), \pi')$. In order to make this construction rigorous, one first considers the restrictions $(\Pi_{|[n]}(t))$ as in Proposition 1.10, since the measure $\nu(dx) := x^{-2}\Lambda(dx)$ can have an infinite mass.

1.4.2 M-coalescents

The distinguished exchangeable coalescents such that when a coagulation occurs all the blocks involved merge as a single block are called M-coalescents. We specify their laws by two finite measures on $[0,1]$, and study their generators in the same fashion as those of Λ-coalescents.

Definition 1.14 *When a distinguished coagulation measure μ is carried by the set of simple distinguished partitions (with only one block non empty nor singleton), the distinguished coalescent Π is said to be simple. Define the following restricted measures :*

$$\nu_0 = \nu 1_{\{s\in\mathcal{P}^0_m; s=(s_0,0,\dots)\}} \text{ and } \nu_1 = \nu 1_{\{s\in\mathcal{P}^0_m; s=(0,s_1,0,\dots)\}}.$$

By a slight abuse of notation, ν_0 and ν_1 can be viewed as two measures on $[0,1]$ such that $\int_0^1 s_0\nu_0(ds_0) < \infty$ and $\int_0^1 s_1^2\nu_1(ds_1) < \infty$. Denote $\rho_{\nu_0}(d\pi) = \rho_\nu(d\pi; |\pi_0| \geq 0)$ and $\rho_{\nu_1}(d\pi) = \rho_\nu(d\pi; |\pi_0| = 0)$. Theorem 1.12 yields

$$\mu(d\pi) = c_0\mu_0^K(d\pi) + \rho_{\nu_0}(d\pi) + c_1\mu_1^K(d\pi) + \rho_{\nu_1}(d\pi).$$

We define the finite measures $\Lambda_0(dx) := x\nu_0(dx) + c_0\delta_0$, and $\Lambda_1(dx) := x^2\nu_1(dx) + c_1\delta_0$. The law of a simple distinguished coalescent is then characterized by $M = (\Lambda_0, \Lambda_1)$, and we call this subclass the M-coalescents.

As already mentioned in Section 3, in the most cases, the restriction to $\mathbb{N}$ of a distinguished coalescent is not Markovian. Let Π be a M-coalescent with for instance $\Lambda_0 = \delta_0$ and $\Lambda_1(dx) = dx$ (the Lebesgue measure). To locate the distinguished block in $\Pi_{|\mathbb{N}}$, we may locate a binary coagulation before time t (all other mergers involve an infinite number of blocks). The restricted process $\Pi_{|\mathbb{N}}$ is then not Markovian.

The explicit Poissonian construction of Proposition 1.10 can now be interpreted in the same way as the one of Λ-coalescent, recalled in Section 4.1. When $\Lambda_0(\{0\}) = \Lambda_1(\{0\}) = 0$, the M-coalescent associated can be constructed via two Poisson Point Processes N_0 and N_1 on $\mathbb{R}_+ \times (0,1]$ with intensities $dt \otimes \nu_0(dx)$ and $dt \otimes \nu_1(dx)$, where $\nu_0(dx) = x^{-1}\Lambda_0(dx)$ and $\nu_1(dx) = x^{-2}\Lambda_1(dx)$:

- At an atom (t_i, x_i) of N_1, flip a coin with probability of "heads" x_i for each block not containing 0. All blocks flipping "heads" are merged immediately in one block as in the Proposition 1.10.
- At an atom (t_i, x_i) of N_0, flip a coin with probability of "heads" x_i for each block not containing 0. All blocks flipping "heads" coagulate immediately with the distinguished block.

This construction is exactly the one obtained when we coagulate the partition at $t-$ with a simple exchangeable distinguished partition π' where the non trivial block is constituted by indexes of "heads". Thus, we construct the M-coalescent in the same way that Λ-coalescent in Section 4.1.

We investigate jump rates of a M-coalescent $(\Pi(t))_{t\geq 0}$. Thanks to the simple distinguished paint-box structure, we compute explicitly the jump rates of the restriction of Π. Let $\pi \in \mathcal{P}^0_n$ be simple, $q_\pi = \mu(\mathcal{P}^0_\infty, \pi)$:

- For every $2 \leq k \leq n$, if π has one block not containing 0 with k elements, then

$$q_\pi = \lambda_{n,k} := \int_0^1 x^{k-2}(1-x)^{n-k}\Lambda_1(dx).$$

– For every $1 \leq k \leq n$, if the distinguished block of π has $k+1$ elements (counting 0), then
$$q_\pi = r_{n,k} := \int_0^1 y^{k-1}(1-y)^{n-k}\Lambda_0(dy).$$

Let $\pi \in \mathcal{P}_p^0$ with b blocks without 0 and F any function defined on $\mathcal{P}_p^0$. The generator of $\Pi_{|[p]}$ is

$$\mathcal{L}^*F(\pi) = \sum_{I\subset\{1,\dots,b\},|I|\geq 2} \lambda_{b,|I|}(F(c_I\pi) - F(\pi)) + \sum_{J\subset\{1,\dots,b\},|J|\geq 1} r_{b,|J|}(F(c^J\pi) - F(\pi)),$$

with $c_I\pi = coag(\pi, \{\{0\},\{1\},...,\{I\},...\})$ and $c^J\pi = coag(\pi,\{\{0\}\cup\{J\},\{.\},...,\{.\}\})$.

1.4.3 Coming down from infinity for M-coalescents

Let Λ be a finite measure on $[0,1]$ and Π be a Λ-coalescent. Pitman [Pit99] showed that if $\Lambda(\{1\}) = 0$, only the following two types of behavior are possible, either $\mathbb{P}$[For all $t > 0$, $\Pi(t)$ has infinitely many blocks] $= 1$, or $\mathbb{P}$[For all $t > 0$, $\Pi(t)$ has only finitely many blocks] $= 1$. In the second case, the process Π is said to come down from infinity. For instance, Kingman's coalescent comes down from infinity, while if $\Lambda(dx) = dx$, then the corresponding Λ-coalescent (called Bolthausen-Sznitman coalescent) does not come down from infinity. A necessary and sufficient condition for a Λ-coalescent to come down from infinity was given by Schweinsberg in [Sch00b]. Define

$$\phi(n) = \sum_{k=2}^{n}(k-1)\binom{n}{k}\lambda_{n,k}$$

with $\lambda_{n,k} = \int_0^1 x^{k-2}(1-x)^{n-k}\Lambda(dx)$. The Λ-coalescent comes down from infinity if and only if
$$\sum_{n=2}^{\infty}\frac{1}{\phi(n)} < \infty.$$

Define $\psi_\Lambda(q) := \int_{[0,1]}(e^{-qx} - 1 + qx)x^{-2}\Lambda(dx)$, Bertoin and Le Gall observed, in [BLG06] the following equivalence

$$\sum_{n=2}^{\infty}\frac{1}{\phi(n)} < \infty \Longleftrightarrow \int_a^{\infty}\frac{dq}{\psi_\Lambda(q)} < \infty$$

where the right-hand side holds for some $a > 0$ (and then necessary for all). This equivalence is explained in a probabilistic way by Berestycki et al. in [BBL].

As for the Λ-coalescent, if the M-coalescent comes down from infinity, it does immediately :

Proposition 1.15 *Let $(\Pi(t))_{t\geq 0}$ be a M-coalescent, with Λ_0 and Λ_1 without mass at 1. We denote by T its time of coming down from infinity : $T = \inf\{t > 0, \#\Pi(t) < \infty\}$. We have a.s $T = 0$ or $T = \infty$.*

Proof. See Lemma 31 of [Sch00a]. The proof is given in the general context of any distinguished coalescent in Chapter 4. □
We stress that when $\Lambda_0 + \Lambda_1$ has a mass at 1, the M-coalescent comes down from infinity. Indeed, by the Poisson construction, in an exponential time τ of parameter $(\Lambda_0+\Lambda_1)(\{1\})$, the Poisson measure N has an atom π such that $\pi_0 = \mathbb{Z}_+$ or $\pi_1 = \mathbb{N}$. Thus for large t, the

process $\Pi(t)$ has just one block.

It remains to focus on the case where $\Lambda_0 + \Lambda_1$ has no mass at 1. Intuitively, when the genuine Λ_1-coalescent comes down, all blocks merged in one in an almost surely finite time. On the one hand, we can think that the (Λ_0, Λ_1)-coalescent has more jumps and coagulates all its blocks faster. On the other hand, the perturbation due to the coagulation with the distinguished block on the general term of the sum, studied initially by Schweinsberg in [Sch00b], is not sufficient to induce its convergence and so the coming down. The (Λ_0, Λ_1)-coalescent comes down from infinity if and only if the Λ_1-coalescent comes down :

Theorem 1.16 *The (Λ_0, Λ_1)-coalescent comes down from infinity if and only if*

$$\sum_{n=2}^{\infty} \frac{1}{\phi_1(n)} < \infty$$

where $\phi_1(n) = \sum_{k=2}^{n} (k-1)\binom{n}{k}\lambda_{n,k}$ and $\lambda_{n,k}$ as in Section 4.2.

The proof requires rather technical arguments and is given in the Section 6.

1.5 M-coalescents and generalized Fleming-Viot processes with immigration

In the final section, we are interested in a population model which has exactly a genealogy given by a M-coalescent. A powerful method to study simultaneously the population model and its genealogy is to define some stochastic flows as Bertoin and Le Gall in [BLG03]. A process valued in the space of *probability measures* on $[0,1]$, $(\rho_t, t \geq 0)$ is embedded in the flow. The atoms of the random probability ρ_t represent the current types frequencies in the population at time t. Moreover, ρ_t has a distinguished atom at 0 representing the fraction of immigrants in the population. This process will be called M-generalized Fleming-Viot processes with immigration. Following [BLG03], we begin by establishing a correspondence between some stochastic flows and M-coalescents.

1.5.1 Stochastic flows of distinguished bridges

By assumption, at any time the families describing the population form a distinguished exchangeable partition. Theorem 1.5 ensures that it has a distinguished paint-box structure. We have to study some random functions called distinguished bridges.

Distinguished bridges and exchangeable distinguished partitions

Considering the underlying law on $[0,1]$ associated with a $\mathbf{s}$-distinguished paint-box (see definition 1.3), we introduce the distinguished bridges defined by

$$b_{\mathbf{s}}(r) = s_0 + \sum_{i=1}^{\infty} s_i 1_{\{V_i \leq r\}} + \delta r$$

where $\mathbf{s}$ is a distinguished mass-partition and $(V_i)_{i\geq 1}$ a sequence of independent uniform variables. Let $U_0 = 0$ and $(U_i)_{i\geq 1}$ be an independent sequence of i.i.d uniform variables. The partition given by $i \sim j$ if and only if $b_{\mathbf{s}}^{-1}(U_i) = b_{\mathbf{s}}^{-1}(U_j)$ is exactly the $\mathbf{s}$-distinguished paint-box. When $s_0 = 0$, the bridge encodes a paint-box partition with no distinguished

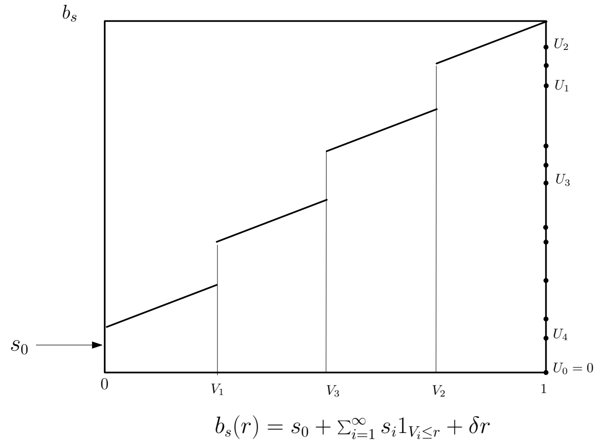


FIGURE 1.1 – Distinguished bridge

block.
Focusing on M-coalescents, we only need to focus on two types of distinguished bridges :

- Bridges with distinguished mass-partition $(0, x, 0, ...)$: $b_{0,x}(r) = x1_{V \leq r} + r(1-x)$.
- Bridges with distinguished mass-partition $(x, 0, 0, ...)$: $b_{x,0}(r) = x + r(1-x)$.

To be concise, we shall work directly with distinguished bridges of the form

$$b_{y,x}(r) = y + x1_{V \leq r} + r(1-x-y).$$

The following lemma relates the composition of distinguished bridges to the coagulation of simple distinguished partitions.

Lemma 1.17 *Let $x, x', y, y' \in [0,1]$ with $x+y, x'+y' \leq 1$ and $b_{y,x}$, $b_{y',x'}$ two independent distinguished bridges.*
Define the partition

$$\pi : i \sim j \text{ if and only if } b_{y,x}^{-1}(U_i) = b_{y,x}^{-1}(U_j).$$

We stress that π has at most two non-trivial blocks (with one containing 0).

(i) For $i \geq 1$, we set $U'_i := b_{y,x}^{-1}(U_j)$, $\forall j \in \pi_i$. The variables $(U'_i)_{i\geq 1}$ are i.i.d uniform, independent of π.

(ii) Let π' be the partition constructed from $b_{y',x'}$ and $(U'_i)_{i\geq 1}$, we denote by π'' the partition such that

$$i \sim j \text{ if and only if } b_{y,x}^{-1} \circ b_{y',x'}^{-1}(U_i) = b_{y,x}^{-1} \circ b_{y',x'}^{-1}(U_j).$$

We have the identity $\pi'' = coag(\pi, \pi')$.

Proof. The proof is an easy adaptation of Lemma 4.8 of [Ber06]. We provide some details. The right-continuous inverse of $b_{y,x}$ is defined by $b_{y,x}^{-1}(r) := \inf\{v \in [0,1]; b_{y,x}(v) > r\}$. An easy calculation gives

$$b_{y,x}^{-1}(r) = \begin{cases} 0 \quad \forall r \in [0, y] \\ \frac{r-y}{(1-x-y)} \quad \forall r \in [y, y + (1-x-y)V] \\ V \quad \forall r \in [y + (1-x-y)V, x + y + (1-x-y)V] \\ \frac{r-[x+y]}{(1-x-y)} \quad \forall r \in [(1-x-y)V + x + y, 1] \end{cases}$$

(i) : Let $n_1 < ... < n_k$ be k integers of $\{1, ..., n\}$. Conditionally on $V = v \in]0, 1[$, the probability of the event "$\{n_1\}, ..., \{n_k\}$ are blocks of π" is

$$\mathbb{P}[U_{n_j} \notin]0, y[\cup](1-x-y)v + y, x + y + (1-x-y)v[, \forall i \leq k] = (1-x-y)^k.$$

This quantity does not depend of v. On this event, $U_{n_1}, ..., U_{n_k}$ are k independent uniform variables on $[y, y + (1-x-y)v] \cup [(1-x-y)v + x + y, 1]$ and so, from the calculus of the inverse : $b_{y,x}^{-1}(U_{n_1}), ..., b_{y,x}^{-1}(U_{n_k})$ are k independent uniform variables.

Let $k \geq 1$ and $\gamma \in \mathcal{P}_n^0$ be such that $\{n_1\}, ..., \{n_k\}$ are exactly the singleton-blocks not containing 0 and $\#\gamma_1 \geq 2$. By exchangeability of π, we assume that π_1 is the non-trivial block not containing 0. We have $U_1' = V$, and from the previous argument :

$$\mathbb{P}[\pi_{|[n]} = \gamma, U_1' \in du_1, ..., U_{k+1}' \in du_{k+1}] = \mathbb{P}[\pi_{|[n]} = \gamma | U_1' \in du_1] du_1 du_2 ... du_{k+1}.$$

The quantity $\mathbb{P}[\pi_{|[n]} = \gamma | V \in dv]$ does not depend on v and is equal to $\mathbb{P}[\pi_{|[n]} = \gamma]$.

It remains to consider the case of $\pi_{|[n]} = \gamma$, with $\#\gamma_i = 1 \ \forall i \geq 1$ which means there is at most one index i such that $U_i \in]y + [1-(x+y)]V, x + y + [1-(x+y)]V[$, that is $b_{x,y}^{-1}(U_i) = U_i' = V$. Let $k = n - (\#\gamma_0 - 1)$ (that is the number of blocks of γ without counting γ_0), by a simple calculation, we get

$$\begin{aligned} \mathbb{P}[\pi_{|[n]} = \gamma, U_1' \in du_1, ..., U_k' \in du_k] = \ & \mathbb{P}[\exists! i \in [k], b_{x,y}^{-1}(U_i) = V, \pi_{|[n]} = \gamma, U_1' \in du_1, ..., U_k' \in du_k] \\ & + \mathbb{P}[\forall i \in [k], b_{x,y}^{-1}(U_i) \neq V, \pi_{|[n]} = \gamma, U_1' \in du_1, ..., U_k' \in du_k]. \end{aligned}$$

On the one hand, we have

$$\begin{aligned} \mathbb{P}[\exists! i \in [k], & b_{x,y}^{-1}(U_i) = V, \pi_{|[n]} = \gamma, U_1' \in du_1, ..., U_k' \in du_k] \\ & = \sum_{i=1}^{k} \mathbb{P}[b_{x,y}^{-1}(U_i) = V, \ \forall j \neq i \ b_{x,y}^{-1}(U_j) \notin \{0, V\}, \pi_{|[n]} = \gamma | V \in du_i] \mathbb{P}[U_1' \in du_1, ..., U_k' \in du_k] \\ & = y^{\#\gamma_0 - 1} kx(1-x-y)^{k-1} du_1 ... du_k. \end{aligned}$$

On the other hand,

$$\mathbb{P}[\forall i \in [k], \ b_{x,y}^{-1}(U_i) \notin \{0, V\}, \pi_{|[n]} = \gamma, U_1' \in du_1, ..., U_k' \in du_k] = y^{\#\gamma_0 - 1}(1-x-y)^k du_1 ... du_k.$$

Therefore

$$\mathbb{P}[\pi_{|[n]} = \gamma, U_1' \in du_1, ..., U_k' \in du_k] = y^{\#\gamma_0 - 1} \left(kx(1-x-y)^{k-1} + (1-x-y)^k \right) du_1 ... du_k.$$

An easy calculation (using the paintbox structure) yields

$$\mathbb{P}[\pi_{|[n]} = \gamma] = y^{\#\gamma_0 - 1} \left(kx(1-x-y)^{k-1} + (1-x-y)^k \right),$$

and so the claim is proved.

Proof of *(ii)* : Let i and j two integers. Let k and l in $\mathbb{Z}_+$ be the respective indices of the blocks of π which contain i and j, that is $i \in \pi_k$ and $j \in \pi_l$. We define

$$U'_k := b_{y,x}^{-1}(U_i) \text{ for all } i \in \pi_k \text{ and } U'_l := b_{y,x}^{-1}(U_j) \text{ for all } j \in \pi_l.$$

The distinguished partition π' is defined by $k \sim l$ if and only if $b_{y',x'}^{-1}(U'_k) = b_{y',x'}^{-1}(U'_l)$. The integers i and j are in the same block of π'' implies that l and k are in the same block of π', and by *coag* operator's definition i and j are in the same block of $coag(\pi, \pi')$. □

This lemma is the key observation in order to associate a stochastic flow of distinguished bridges, as defined below, with M-coalescents.

Definition 1.18 *A flow of distinguished bridges is a collection* $(B_{s,t}, -\infty < s \leq t < \infty)$ *of distinguished bridges such that*

- $\forall s < t < u, B_{s,u} = B_{s,t} \circ B_{t,u}$ *a.s.*
- *The law of* $B_{s,t}$ *depends only on* $t - s$*, and for any* $s_1 < ... < s_n$*,* $B_{s_1,s_2}, ..., B_{s_{n-1},s_n}$ *are independent.*
- $B_{0,0}$ *=Id,* $B_{0,t} \to$ *Id in probability when* $t \to 0$*.*

Poissonian construction of distinguished flows encoding M-coalescent

Let M^0 and M^1 be two independent Poissonian measures on $\mathbb{R} \times [0,1]$ with intensities $dt \otimes \nu_0(dx)$ and $dt \otimes \nu_1(dx)$. We suppose $\nu_0([0,1]) + \nu_1([0,1]) < \infty$ so that $N_{s,t} := (M^0 + M^1)(]s,t] \times [0,1])$ is finite and $(N_t) = (N_{0,t})_{t \geq 0}$ is a Poisson process. Let (t_i^0, x_i^0), (t_i^1, x_i^1) be the atoms of M^0 and M^1 in $]s,t] \times [0,1]$, we define $B_{s,t} = b_{x_1} \circ ... \circ b_{x_K}$ where $K = N_{s,t}$ and where b_x denotes $b_{0,x}$ or $b_{x,0}$ depending on whether x is an atom of M^0 or M^1. From the independence of M^0 and M^1 and the independence of $M^i(A)$ and $M^i(B)$, $i = 0,1$ for A and B disjoint, $(B_{s,t})_{s \leq t}$ is a flow in the sense of definition 1.18.

Proposition 1.19 *The process* $(\Pi(t), t \geq 0)$ *:* $i \sim j \Leftrightarrow B_{0,t}^{-1}(U_i) = B_{0,t}^{-1}(U_j)$ *where* $B_{0,t} = b_{x_1} \circ ... \circ b_{x_{N_t}}$ *is a* M*-coalescent with* $M = (x\nu_0(dx), x^2\nu_1(dx))$*.*

Proof. Lemma 1.17 implies that the process $(\Pi(t), t \geq 0)$ corresponds to that built explicitly in Proposition 1.10. □

The next result defines stochastic flows for general measures ν_0 and ν_1 on $[0,1]$.

Theorem 1.20 *Let* (ν_0^n) *and* (ν_1^n) *be two sequences of finite measures on* $[0,1]$*. We call* $(B_{s,t}^{(n)}, -\infty < s \leq t < \infty)$ *the associated flow of bridges. Assume the weak convergences of* $\Lambda_1^n(dx) := x^2\nu_1^n(dx)$ *to* $\Lambda_1(dx) := c_1\delta_0(dx) + x^2\nu_1(dx)$ *and* $\Lambda_0^n(dy) := y\nu_0^n(dy)$ *to* $\Lambda_0(dy) := c_0\delta_0 + y\mu(dy)$*. We get*

- $(B_{s,t}^{(n)}, \infty < s \leq t < \infty)$ *converges, in the sense of convergence of finite-dimensional distributions, to* $(B_{s,t}, \infty < s \leq t < \infty)$ *a stochastic flow.*
- *The process* $(\Pi(t), t \geq 0)$ *defined by* $\Pi(t) : i \sim j \Leftrightarrow B_{0,t}^{-1}(U_i) = B_{0,t}^{-1}(U_j)$ *is a* M*-coalescent with rates* (Λ_0, Λ_1)*.*

Proof. We denote by $\Pi^{(n)}(s,t)$ the random partition encoded by $B_{s,t}^{(n)}$. Under the previous assumptions on ν_0 and ν_1, the jump rates $\lambda_{b,k}^{(n)} = \int_0^1 x^k(1-x)^{b-k}\nu_1^n(dx)$ and $r_{b,k}^{(n)} = \int_0^1 y^k(1-y)^{b-k}\nu_0^n(dy)$ converge :

$$\lambda_{b,k}^{(n)} \underset{n\to\infty}{\to} c_1 1_{k=2} + \int_0^1 x^k(1-x)^{b-k}\nu_1(dx) := \lambda_{b,k}.$$

$$r_{b,k}^{(n)} \underset{n\to\infty}{\to} c_0 1_{k=1} + \int_0^1 x^k(1-x)^{b-k}\nu_0(dx) := r_{b,k}.$$

The sequence of Markov chains $(\Pi^{(n)}_{|[k]}(s,t))_{t\geq s}$ converges in the sense of finite-dimensional distributions to a distinguished coalescent chain, say $\Pi_{|[k]}(s,t)_{t\geq s}$. By compatibility, this implies the convergence of finite-dimensional distributions of $(\Pi^{(n)}(s,t))_{n\geq 1}$ to $\Pi(s,t)$. According to Proposition 2.9 and Lemma 4.7 in [Ber06] (which are easily adapted to our setting), we obtain the convergence of the distinguished mass-partitions $|\Pi^{(n)}(s,t)|^{\downarrow} \underset{n\to\infty}{\to} |\Pi(s,t)|^{\downarrow}$ and the convergence of the bridge $B_{s,t}^{(n)}$ (which has jumps of size $|\Pi^{(n)}(s,t)|^{\downarrow}$) to a bridge $B_{s,t}$ (which has jumps of size $|\Pi(s,t)|^{\downarrow}$) for all $s,t\geq 0$ fixed. Thanks to the independence of $B^{(n)}_{s_1,s_2},...,B^{(n)}_{s_{k-1},s_k}$ for any $s_1<...<s_k$ and the flow property ($B^{(n)}_{s,t}\circ B^{(n)}_{t,u}=B^{(n)}_{s,u}$), the one-dimensional convergence in distribution readily extends to finite-dimensional distribution. The existence of the flow B is ensured by Kolmogorov's Extension Theorem.□

Remark 3 *As mentioned in Section 2, we could define coalescents with several distinguished blocks. In particular, considering distinguished bridges which jump at* 0 *and* 1*, we get a flow coding a coalescent with two distinguished blocks and a population with two immigration sources.*

The composition of two distinguished bridges may be interpreted as the succession of two events (reproduction or immigration) in the population. A duality method provides a continuous population model.

1.5.2 The dual distinguished flow and a population model with immigration

In the same spirit of [Ber06] and [BLG03], we interpret the dual flow, $(\hat{B}_{s,t}) := (B_{-t,-s})$ in terms of a natural model population on $[0,1[$ with fixed size 1. We denote by $\rho_t(dr)$, the random Stieltjes measure of $\hat{B}_{0,t}$: $\rho_t = d\hat{B}_{0,t}$, it defines a Markov process with values in the space of probability measures on $[0,1[$ (denoted by $\mathcal{M}_1$). We may think of $\rho_t(dr)$ and $\rho_t(\{0\})$ respectively as the size of the progeny at time t of the fraction dr of the initial population, and as the size at time t of the immigrants descendants.
The cocycle identity $\hat{B}_{t,u}\circ\hat{B}_{s,t}=\hat{B}_{s,u}$ ensures that $(\rho_t, t\geq 0)$ is a continuous-time Markov chain with the following dynamics, whenever the measures ν_0 and ν_1 are finite : if t is a jump time for $(\rho_t, t\geq 0)$, then the conditional law of ρ_t given ρ_{t-} is that of

- $(1-X)\rho_{t-}+X\delta_U$, if t is an atom of M^1, where X is distributed as $\nu_1(.)/\nu_1([0,1])$ and U as ρ_{t-}.
- $(1-Y)\rho_{t-}+Y\delta_0$, if t is an atom of M^0, where Y is distributed as $\nu_0(.)/\nu_0([0,1])$.

At a reproduction time (meaning an atom of M^1) an individual picked at random in the population at generation $t-$ generates a proportion X of the population at time t, as for the genuine generalized Fleming-Viot. At an immigration time (meaning an atom of M^0) the individual 0 at the time $t-$ generates a proportion Y of the population at time t. In both cases, the rest of the population at time $t-$ is reduced by a factor $1-X$ or $1-Y$ so that, at time t, the total size is still 1. We call this measure-valued process a generalized Fleming-Viot process with immigration (GFVI). The genealogy of this population (which is identified as $[0,1]$) coincides with an M-coalescent. Plainly, the generator of $(\rho_t, t\geq 0)$

is

$$\mathcal{L}G(\rho) = \int \nu_1(dx) \int \rho(da)[G((1-x)\rho + x\delta_a) - G(\rho)] + \int \nu_0(dy)[G((1-y)\rho + y\delta_0) - G(\rho)].$$

Thus, for any bounded function G on $\mathcal{M}_1$, the space of probability measures on $[0,1]$

$$\begin{aligned} G(\rho_t) \quad - \quad & \int_0^t ds \int \nu_1(dx) \int \rho_s(da)[G((1-x)\rho_s + x\delta_a) - G(\rho_s)] \\ & - \int_0^t ds \int \nu_0(dy)[G((1-y)\rho_s + y\delta_0) - G(\rho_s)] \end{aligned}$$

is a martingale. Considering the functions of the form

$$G_f : \rho \in \mathcal{M}_1 \mapsto \int_{[0,1]^p} f(x_1, ..., x_p)\rho(dx_1)...\rho(dx_p) = \langle f, \rho^{\otimes p} \rangle,$$

for f a continuous function on $[0,1]^p$, we generalize in the following lemma this result for infinite measures.

Lemma 1.21 *Assume that ν_0 and ν_1 have infinite masses and c_0, c_1 are zero, we define the operator $\mathcal{L}$, acting on functions of the type G_f, by*

$$\mathcal{L}G_f(\rho) = \int \nu_1(dx) \int \rho(da)[G_f((1-x)\rho + x\delta_a) - G_f(\rho)] + \int \nu_0(dy)[G_f((1-y)\rho + y\delta_0) - G_f(\rho)].$$

The process $G_f(\rho_t) - \int_0^t \mathcal{L}G_f(\rho_s)ds$ is a martingale.

Proof. Consider two sequences (ν_1^n) and (ν_0^n) of finite measures on $[0,1[$, suppose that $\Lambda_1^n(dx) := x^2\nu_1^n(dx)$ and $\Lambda_0^n(dx) := x\nu_0^n(dx)$ weakly converge to some finite measures $\Lambda_1(dx)$ and $\Lambda_0(dx)$. For $G_f(\rho) = \prod_i^p \langle \phi_i, \rho \rangle$, Bertoin and Le Gall obtain in [BLG03] :

$$\begin{aligned} & \int \nu_1^n(dx) \int \rho(da)[G_f((1-x)\rho + x\delta_a) - G_f(\rho)] \\ & \qquad = \sum_{\substack{I \subset \{1,...,p\} \\ |I| \geq 2}} \lambda_{p,|I|}^n \int \rho(dx_1)...\rho(dx_p)[f(x_1^I, ..., x_p^I) - f(x_1, ..., x_p)]. \end{aligned}$$

With $(x_1^I, ..., x_p^I) = (y_1, ..., y_p)$ where for all $i \in I$, $y_i = x_{\inf I}$ and the values y_i, $i \notin I$ listed in the order of $\{1, ..., p\} \setminus I$ are the numbers $x_1, .., x_{\inf I - 1}, x_{\inf I + 1}, ..., x_{p - |I| + 1}$.

The assumption on ν_1 ensures that the right side converges to

$$\sum_{I \subset \{1,...,p\}; |I| \geq 2} \lambda_{p,|I|} \int \rho(dx_1)...\rho(dx_p)[f(x_1^I, ..., x_p^I) - f(x_1, ..., x_p)].$$

The reasoning is similar for the study of the "immigration" part. We establish

$$\begin{aligned} & \int \nu_0^n(dy)[G_f((1-y)\rho + y\delta_0) - G_f(\rho)] \\ & \qquad = \sum_{J \subset \{1,...,p\}; |J| \geq 1} r_{p,|J|}^n \int \rho(dx_1)...\rho(dx_p)[f(x_1^0, ..., x_p^0) - f(x_1, ..., x_p)] \end{aligned}$$

with $(x_1^0, ..., x_p^0) = (z_1, ..., z_p)$ where for all $i \in J$, $z_i = 0$ and the values z_i, $i \notin J$ listed in the order of $\{1, ..., p\} \setminus J$ are the numbers $x_1, .., x_{\inf J - 1}, x_{\inf J + 1}, ..., x_{p - |J|}$.

An easy calculation gives

$$G_f((1-y)\rho+y\delta_0)=\prod_{i=1}^{p}[(1-y)\langle\phi_i,\rho\rangle+y\phi_i(0)]=\sum_{J\subset\{1,\dots,p\}}(1-y)^{p-|J|}y^{|J|}\prod_{j\notin J}\langle\rho,\phi_j\rangle\prod_{j\in J}\phi_j(0)$$

and then from the following identity

$$\prod_{j\notin J}\langle\rho,\phi_j\rangle\prod_{j\in J}\phi_j(0)=\int_{[0,1]^p}f(x_1^0,\dots,x_p^0)\rho(dx_1)\dots\rho(dx_p)$$

it follows that

$$\int\nu_0^n(dy)G((1-y)\rho+y\delta_0)=\sum_{J\subset\{1,\dots,p\};|J|\geq 1}r^n_{p,|J|}\int\rho(dx_1)\dots\rho(dx_p)f(x_1^0,\dots,x_p^0).$$

Moreover, the right hand side converges to

$$\sum_{J\subset\{1,\dots,p\};|J|\geq 1}r_{p,|J|}\int\rho(dx_1)\dots\rho(dx_p)f(x_1^0,\dots,x_p^0),$$

by passing to the limit in $n\to\infty$, it follows that for $f(x_1,\dots,x_p)=\prod_i^p\phi_i(x_i)$, the process

$$M_f(t):=G_f(\rho_t)-\int_0^t LG_f(\rho_s)ds$$

is a martingale where L is the operator defined by

$$\begin{aligned}LG_f(\rho)=&\sum_{I\subset\{1,\dots,p\};|I|\geq 2}\lambda_{p,|I|}\int\rho(dx_1)\dots\rho(dx_p)[f(x_1^I,\dots,x_p^I)-f(x_1,\dots,x_p)]\\&+\sum_{J\subset\{1,\dots,p\};|J|\geq 1}r_{p,|J|}\int\rho(dx_1)\dots\rho(dx_p)[f(x_1^0,\dots,x_p^0)-f(x_1,\dots,x_p)].\end{aligned}$$

Since any continuous function on $[0,1]^p$ is the uniform limit of linear combinations of functions of the previous type, we easily conclude that M_f is a martingale for any continuous function on $[0,1]^p$. The statement claims that when $c_0=c_1=0$ the generator has an integral form as the one obtained for finite measures. We assume now that c_0 and c_1 are zero. Let $A_1,\dots,A_p$ i.i.d variables distributed as ρ, and $x,y\in[0,1]$. Let (β_j), (β'_j) be two sequences of Bernoulli variables of parameters x and y. We set $I:=\{j,\beta_j=1\}$ and $J:=\{j,\beta'_j=1\}$. Let f be a continuous function on $[0,1]^p$, for $G_f(\rho)=\langle\rho^{\otimes p},f\rangle$, it is readily checked (see [BBC+05]) that

$$\begin{aligned}&\int\rho(da)[G_f((1-x)\rho+x\delta_a)-G_f(\rho)]=\mathbb{E}[f(A_1^J\dots,A_p^J)]-\mathbb{E}[f(A_1,\dots,A_p)]\\&=\sum_{I\subset\{1,\dots,p\},|I|\geq 2}x^{|I|}(1-x)^{p-|I|}\int\rho(dx_1)\dots\rho(dx_p)(f(x_1^I,\dots,x_p^I)-f(x_1,\dots,x_p)),\end{aligned}$$

$$\begin{aligned}&G_f((1-y)\rho+y\delta_0)-G_f(\rho)=\mathbb{E}[f(A_1^0\dots,A_p^0)]-\mathbb{E}[f(A_1,\dots,A_p)]\\&=\sum_{J\subset\{1,\dots,p\},|J|\geq 1}y^{|J|}(1-y)^{p-|J|}\int\rho(dx_1)\dots\rho(dx_p)(f(x_1^0,\dots,x_p^0)-f(x_1,\dots,x_p)).\end{aligned}$$

We deduce that the process $(\rho_t,t\geq 0)$ solves the following martingale problem : for any continuous function f on $[0,1]^p$, $G_f(\rho_t)-\int_0^t ds\mathcal{L}G_f(\rho_s)$ is a martingale. □

Proposition 1.22 *The law of the process $(\rho_t, t \geq 0)$ is characterized by the martingale problem of Lemma 1.21, and the operator $\mathcal{L}$ is an extended generator of the process $(\rho_t, t \geq 0)$.*

Proof. We will use the same duality argument as in Bertoin and Le Gall [BLG03]. With their notation, we define a class of functions from $\mathcal{M}_1 \times \mathcal{P}_p^0$ to $\mathbb{R}$

$$\Phi_f : (m, \pi) \in \mathcal{M}_1 \times \mathcal{P}_p^0 \mapsto \int \delta_0(dx_0) m(dx_1) ... m(dx_{\#\pi-1}) f(Y(\pi; x_1, ..., x_{\#\pi-1}))$$

with f a continuous function on $[0,1]^p$ and $Y(\pi; x_1, ..., x_{\#\pi-1}) = (y_1, ..., y_p)$ such that $y_j = x_i$ if $j \in \pi_i$ for any $i \geq 0$.
For a fixed partition π in $\mathcal{P}_n^0$, there exists a function g continuous on $[0,1]^{\#\pi-1}$ with $\mu \mapsto \Phi_f(\mu, \pi) = G_g(\mu)$ and then $\mathcal{L}\Phi_f(\mu, \pi) = \mathcal{L}G_g(\mu)$ is well-defined. We stress that for a fixed measure μ, $\Phi_f(\mu, .)$ is a function on $\mathcal{P}_p^0$. We show the following duality result :

$$\mathbb{E}[\Phi_f(\rho_0, \Pi(t))] = \mathbb{E}[\Phi_f(\rho_t, \Pi_0)].$$

By the cocycle property of the stochastic flow involved, it suffices to focus on process beginning at $\rho_0 = \lambda$.

$$\begin{aligned}
\mathbb{E}_\lambda[\phi_f(\rho_t, 0_{[p]})] &= \mathbb{E}[\int_{[0,1]^{p+1}} \delta_0(dx_0) d\hat{B}_{0,t}(x_1) ... d\hat{B}_{0,t}(x_p) f(x_1, ..., x_p)] \\
&= \mathbb{E}[\int_{[0,1]^{p+1}} \delta_0(dx_0) dx_1 ... dx_p f(\hat{B}_{0,t}^{-1}(x_1), ..., \hat{B}_{0,t}^{-1}(x_p))] \\
&= \mathbb{E}[f(\hat{B}_{0,t}^{-1}(V_1), ..., \hat{B}_{0,t}^{-1}(V_p))]
\end{aligned}$$

where $(V_i, 1 \leq i \leq p)$ are independent and uniformly distributed on $[0,1]$. We define for $1 \leq i \leq \#\Pi_{|[p]}(t) - 1$, $V_i' := \hat{B}_{0,t}^{-1}(V_j)$ for $j \in \Pi_{i|[p]}(t)$. By Lemma 1.17, $(V_i')_{1 \leq i \leq \#\Pi_{|[p]}(t)-1}$ are uniform iid, independent of $\Pi(t)$ where Π is a M-coalescent with rates $(x\nu_0(dx), x^2\nu_1(dx))$. We get,

$$\mathbb{E}_\lambda[\phi_f(\rho_t, 0_{[p]})] = \mathbb{E}[\int_{[0,1]^{p+1}} \delta_0(dx_0) dx_1 ... dx_{\#\Pi_{|[p]}(t)-1} f(y_1, ... y_{\#\Pi^0_{|[p]}(t)-1})]$$

with $y_j = x_i$ if $j \in \Pi_{i|[p]}(t)$.

Thus, we deduce the duality result :

$$\mathbb{E}_\lambda[\Phi_f(\rho_t, 0_{[p]})] = \mathbb{E}[\Phi_f(\lambda, \Pi_{|[p]}(t))] \text{ and then } \mathcal{L}\Phi_f(\mu, \pi) = \mathcal{L}^*\Phi_f(\mu, \pi).$$

From Theorem 4.4.2 in [EK86], this implies uniqueness for the martingale problem, as well as strong Markov property for the solution. The reader may find the statement of this classic theorem in the Annexes (Theorem 4.3).□

Remark 4 *In the case of a standard M-coalescent, ρ_0 is the Lebesgue measure λ, and we have*

$$\rho_t(dr) = |\Pi_0(t)|\delta_0(dr) + \textstyle\sum_{i \geq 1} |\Pi(t)|_i^{\downarrow} \delta_{W_i}(dr) + (1 - \sum_{i \geq 0} |\Pi(t)|_i^{\downarrow}) dr$$

where $(W_i, i \geq 1)$ are i.i.d uniform and independent of $\Pi(t)$ (but depending on t).

The extinction of the initial types corresponds to the absorption of the GFVI process $(\rho_t, t \geq 0)$ at δ_0. Plainly this event occurs if and only if the measure Λ_0 is not the zero measure and the M-coalescent embedded is coming down from infinity. By Theorem 1.16, we know that the coming down from infinity depends only on the measure Λ_1. In terms of the population model, the immigration mechanism, encoded by $\Lambda_0 := c_0\delta_0 + x\nu_0(dx)$, has no impact on the extinction occurrence, provided of course that Λ_0 is not the zero measure.

1.6 Proof of Theorem 1.16

We recall the statement of Theorem 1.16 and give a proof based on martingale arguments.

Theorem 1.16 *The (Λ_0, Λ_1)-coalescent comes down from infinity if and only if*

$$(\Lambda_0 + \Lambda_1)(\{1\}) > 0 \text{ or } \textstyle\sum_{n=2}^{\infty} \frac{1}{\phi_1(n)} < \infty$$

where $\phi_1(n) = \sum_{k=2}^{n}(k-1)\binom{n}{k}\lambda_{n,k}$ *and* $\lambda_{n,k} = \int_0^1 x^{k-2}(1-x)^{n-k}\Lambda_1(dx)$.

For the Λ-coalescents, (in our setting it corresponds to have $\Lambda_0 \equiv 0$), Schweinsberg studied the mean time of "coming down from infinity" and conclude using the Kochen-Stone lemma. The proof we give here is based on martingale arguments. To show that the convergence of the series is sufficient for the coming down from infinity, we need to prove Lemma 1.23 which is close to the one of Proposition 4.9 page 202 in [Ber06]. The necessary part of the proof does not follow Schweinsberg's ideas. Assuming that the coalescent comes down from infinity and the sum is infinite, we will define a supermartingale (thanks to Lemmas 1.24, 1.25 and 1.26) and find a contradiction (Lemma 1.27).

Lemma 1.23 *Let $(\Pi(t), t \geq 0)$ be a (Λ_0, Λ_1)-coalescent where $\Lambda_0 + \Lambda_1$ has no mass at 1. Let us define the fixation time*

$$\zeta := \inf\{t \geq 0, \Pi(t) = \{\mathbb{Z}_+, \emptyset, ...\}\}.$$

Define

$$\phi(n) = \sum_{k=2}^{n}(k-1)\binom{n}{k}\lambda_{n,k} + \Lambda_0([0,1])n,$$

then the expectation of fixation time is bounded by

$$\mathbb{E}[\zeta] \leq \sum_{n=1}^{\infty} 1/\phi(n).$$

As a consequence, if the series in the right-hand side converges, the fixation time is finite with probability one.

Proof. We shall study the process of blocks which do not contain 0 : for all $t > 0$, $\Pi^*(t) := \{\Pi_1(t), ...\}$. This process is not partition-valued. The jump rates of $\#\Pi^*_{|[n]}(t)$ are easily computed : for $2 \leq k \leq l+1$, $\#\Pi^*_{|[n]}$ jumps from l to $l-k+1$ with rate :

$$\binom{l}{k}\lambda_{l,k}1_{k\leq l} + \binom{l}{k-1}r_{l,k-1}.$$

The first term represents coagulation of k blocks not containing 0 in one, the second represents disappearance of $k-1$ blocks (coagulation with Π_0). We get the infinitesimal generator of $\#\Pi^*_{|[n]}$:

$$G^{[n]}f(l) = \sum_{k=2}^{l}\binom{l}{k}\lambda_{l,k}\left[f(l-k+1) - f(l)\right] + \sum_{k=2}^{l+1}\binom{l}{k-1}r_{l,k-1}\left[f(l-k+1) - f(l)\right].$$

We define

$$\phi_1(n) = \sum_{k=2}^{n}(k-1)\binom{n}{k}\lambda_{n,k} \text{ and } \phi_2(n) = \sum_{k=2}^{n+1}(k-1)\binom{n}{k-1}r_{n,k-1}.$$

Using the binomial formula, we get $\phi_2(n) = \Lambda_0([0,1])n$. We remark that ϕ_1 is an increasing function. Setting $\phi(n) = \phi_1(n) + \phi_2(n)$ and assuming the convergence of the sum $\sum_{n=1}^{\infty} 1/\phi(n)$, we define

$$f(l) = \sum_{k=l+1}^{\infty} 1/\phi(k).$$

The map ϕ is increasing, we thus have $f(l-k+1) - f(l) \geq \frac{k-1}{\phi(l)}$ and then

$$G^{[n]}f(l) \geq \sum_{k=2}^{l+1}\left[\binom{l}{k}\lambda_{l,k} + \binom{l}{k-1}r_{l,k-1}\right]\frac{k-1}{\phi(l)} = 1$$

The process $f(\#\Pi^*_{|[n]}(t)) - \int_0^t G^{[n]}f(\#\Pi^*_{|[n]}(s))ds$ is a martingale. The quantity

$$\zeta_n := \inf\{t; \#\Pi^*_{[n]}(t) = 0\}$$

is a finite stopping time. Let $k \geq 1$, applying the optional stopping theorem to the bounded stopping time $\zeta_n \wedge k$, we get :

$$\mathbb{E}[f(\#\Pi^*_{|[n]}(\zeta_n \wedge k))] - \mathbb{E}[\int_0^{\zeta_n \wedge k} G^{[n]}f(\#\Pi^*_{|[n]}(s))ds] = f(n)$$

With the inequality $G^{[n]}f(l) \geq 1$, we deduce that

$$\mathbb{E}[\zeta_n \wedge k] \leq \mathbb{E}[f(\#\Pi^*_{|[n]}(\zeta_n \wedge k))] - f(n).$$

By monotone convergence and Lebesgue's theorem, we have $\mathbb{E}[\zeta_n] \leq f(0) - f(n)$. Passing to the limit in n, we have $\zeta_n \uparrow \zeta_\infty := \inf\{t; \#\Pi(t) = 1\}$ and $f(n) \longrightarrow 0$, thus

$$\mathbb{E}[\zeta_\infty] \leq f(0) = \sum_{k=1}^{\infty} 1/\phi(k).$$

□

By simple series comparisons, we deduce the sufficient part of the theorem. Plainly $\phi(n) \geq \phi_1(n)$, and if the series $\sum_{n=1}^{\infty} 1/\phi_1(n)$ converges, then by Lemma 1.23 the M-coalescent comes down from infinity.

To show that the convergence of the series is necessary for the coming down, we must look more precisely at the behavior of jumps. The next technical lemmas show that when a distinguished coalescent comes down from infinity, there is a finite number of jumps which make decrease by half or more the number of blocks. Lemma 1.25 will allow us to study the process of blocks number before the first of these times. Assuming that the sum $\sum_{n=2}^{\infty}\frac{1}{\phi(n)} = \infty$ is infinite, we will define a supermartingale in Lemma 17 and find a contradiction by applying the optional stopping theorem.

We have already seen that, as for the Λ-coalescent, a way to understand the dynamics of a (Λ_0, Λ_1)-coalescent, when Λ_0, Λ_1 have no mass at 0, is to imagine drawing an infinite sequence of Bernoulli variables at each jump time, with parameter x controlled by the measures $\nu_0(dx) = x^{-1}\Lambda_0(dx)$ and $\nu_1(dx) = x^{-2}\Lambda_1(dx)$. The following technical lemma allows to estimate the chance for a Bernoulli vector to have more than half terms equals to 1.

Lemma 1.24 *Let $(X_1, X_2, ...)$ be independent Bernoulli variables with parameter $x \in [0, 1/4[$. Defining $S_n^{(x)} = X_1 + ... + X_n$, for every n_0, there is the bound*

$$\mathbb{P}[\exists n \geq n_0; S_n^{(x)} > \frac{n}{2}] \leq \frac{\exp(-n_0 f(x))}{1 - \exp(-f(x))}$$

with $f(x) \sim \frac{1}{2} log(1/x)$ when $x \to 0$.

Proof. By Markov inequality, for all $t > 0$,

$$\mathbb{P}[S_n^{(x)} \geq n/2] \leq e^{-nt/2}\mathbb{E}[e^{tS_n^{(x)}}] = \exp(-n[t/2 - \log(xe^t + 1 - x)]).$$

Applying this inequality for $t = \log(1/x)$, we get $\mathbb{P}[S_n^{(x)} \geq n/2] \leq e^{-nf(x)}$ where

$$f(x) = \frac{1}{2}\log(1/x) - \log(2 - x).$$

The function f is non-negative on $]0, \frac{1}{4}[$ and then we obtain the convergence of the geometric sum

$$\mathbb{P}[\exists n \geq n_0; S_n^{(x)} > \frac{n}{2}] \leq \sum_{n \geq n_0} \exp(-nf(x)) = \frac{\exp(-n_0 f(x))}{1 - \exp(-f(x))}.$$

Moreover we have $f(x) \underset{x \to 0^+}{\sim} \frac{1}{2}\log(1/x)$.

Lemma 1.25 *Assume that the M-coalescent comes down from infinity. With probability one, we have*

$$\tau := \inf\{t > 0, \#\Pi(t) < \frac{\#\Pi(t-)}{2}\} > 0.$$

Moreover, if we define $\tau_n := \inf\{t > 0, \#\Pi_{|[n]}(t) \leq \frac{\#\Pi_{|[n]}(t-)}{2}\}$, then the sequence of stopping times τ_n converges to τ almost surely.

Proof. Obviously, binary coagulations play no role in the statement and we may assume that $\Lambda_0(\{0\}) = 0$ and $\Lambda_1(\{0\}) = 0$. Let N be a Poisson measure with intensity $dt \otimes (x^{-1}\Lambda_0(dx) + x^{-2}\Lambda_1(dx))$. Recall the notation in Lemma 1.24. Let $n_0 \geq 4$, we will show that

$$N(\{(t, x); t \leq 1; \exists n \geq n_0; S_n^{(x)} \geq n/2\}) < \infty.$$

By Proposition 1.15, $\#\Pi(\epsilon) < \infty$ a.s for every $\epsilon > 0$. We will then deduce that there is a finite number of jump times before 1 where more than half blocks coagulate. By the Feller property and therefore the regularity of paths of $(\Pi(t))_{t \geq 0}$, $\Pi(0+) = \Pi(0)$ and 0 is not a jump time, then almost surely : $\tau > 0$. Moreover, $\#\Pi(\tau-) < \infty$ and then for all $n \geq \#\Pi(\tau-)$, $\tau_n = \tau$. We deduce that $\tau_n \underset{n \to \infty}{\to} \tau$ almost surely.
By Poissonian calculations, we get

$$\mathbb{E}[N(\{(t, x); t \leq 1; \exists n \geq n_0, S_n^{(x)} > n/2\})] = \int_0^1 (\nu_0 + \nu_1)(dx)\mathbb{P}[\exists n \geq n_0, S_n^{(x)} > n/2].$$

By Lemma 1.24, we get

$$\int_0^1 (\nu_0+\nu_1)(dx)\mathbb{P}[\exists n \geq n_0, S_n^{(x)} > n/2] \leq \int_0^{\frac{1}{4}} (\nu_0+\nu_1)(dx)\frac{\exp(-n_0 f(x))}{1-\exp(-f(x))} + \int_{1/4}^1 (\nu_0+\nu_1)(dx).$$

On the one hand

$$\int_0^{\frac{1}{4}} (\nu_0+\nu_1)(dx)\frac{\exp(-n_0 f(x))}{1-\exp(-f(x))} < \infty$$

because the integrand is bounded by $8x^{\frac{n_0}{2}}$ and $n_0 \geq 4$,

on the other hand

$$\int_{1/4}^1 (\nu_0+\nu_1)(dx) \leq \int_{1/4}^1 x^{-2}x^2(\nu_0+\nu_1)(dx) \leq 16\int_0^1 x^2(\nu_0+\nu_1)(dx) < \infty.$$

This completes the proof.□

Assuming that the coalescent comes down from infinity and that $\sum_{n\geq 1}\frac{1}{\phi(n)} = \infty$, we can define a supermartingale. We will find a contradiction using the optional stopping theorem.
We define the decreasing function :

$$f(n) = \exp(-\sum_{k=1}^{n+1}\frac{1}{\phi(k)})$$

where $\phi(n) = \phi_1(n) + \phi_2(n)$ with $\phi_1(n)$ and $\phi_2(n)$ are defined as in Lemma 1.23.

Lemma 1.26 *There exists a constant $C > 0$ such that for all $n \geq 1$, $(e^{-Ct}f(\#\Pi^*_{|[n]}(t)))_{t\leq\tau_n}$ is a non-negative supermartingale.*

Proof. We recall that the generator of $(\#\Pi^*_{|[n]}(t))_{t\geq 0}$ is

$$G^{[n]}g(l) = \sum_{k=2}^{l+1}\left[\binom{l}{k}1_{k\leq l}\lambda_{l,k} + \binom{l}{k-1}r_{l,k-1}\right][g(l-k+1) - g(l)].$$

Stopping the process at τ_n, the jump times where more half of blocks coagulate are ignored, and the generator of the stopped process is

$$A^{[n]}g(l) = \sum_{k=2}^{l/2+1}\left[\binom{l}{k}1_{k\leq l/2}\lambda_{l,k} + \binom{l}{k-1}r_{l,k-1}\right][g(l-k+1) - g(l)].$$

We set $\Psi(n) = \int_0^1 (e^{-nx} - 1 + nx)\nu_1(dx)$, an easy verification allows to claim the existence of $c > 0$ such that $c\Psi(q) \leq \phi_1(q) \leq \Psi(q)$ (see remark p170 in[BLG06]), moreover

$$\frac{\Psi(q)}{q} = \int_0^1 (1-e^{-qx})\nu_1([x,1[)dx \underset{q\to\infty}{\rightarrow} \int_0^1 x\nu_1(dx) > 0.$$

Plainly, $h(q) = \Psi(q)/q$ is a concave function, so $h(q/2) \geq h(q)/2$ and $\Psi(q/2) \geq \Psi(q)/4$.
Let us compute

$$A^{[n]}f(l) = \sum_{k=2}^{l/2+1}\left[\binom{l}{k}1_{k\leq l/2}\lambda_{l,k} + \binom{l}{k-1}r_{l,k-1}\right]f(l)[\exp(\sum_{l-k+2}^{l}\frac{1}{\phi(j)}) - 1].$$

We have

$$\sum_{l-k+2}^{l} \frac{1}{\phi(j)} \leq \frac{k-1}{\phi(l-k+2)} \leq \frac{k-1}{\phi(l/2)} \quad \forall k \leq \frac{l}{2}+1$$

and $e^x - 1 \leq cx$ for small x, then for large l

$$A^{[n]}f(l) \leq cf(l) \sum_{k=1}^{l/2+1} \left[\binom{l}{k} \lambda_{l,k} + \binom{l}{k-1} r_{l,k-1} \right] \frac{k-1}{\phi(l/2)}.$$

Thus,

$$A^{[n]}f(l) \leq cf(l) \frac{\phi(l)}{\phi(l/2)}.$$

By definition $\phi(l) = \phi_1(l) + \Lambda_0([0,1])l$. Moreover $l/\phi_1(l)$ is bounded (it converges to $(\int_{[0,1]} x\nu_1(dx))^{-1}$) and from the inequalities : $c\Psi(l)/l \leq \phi_1(l)/l \leq \Psi(l)/l$ and $\Psi(l/2) \geq \Psi(l)/4$, we deduce that for some constant $C > 0$

$$\frac{\phi(l)}{\phi(l/2)} \leq \frac{\phi_1(l) + \Lambda([0,1])l}{\phi_1(l/2)} = \frac{\phi_1(l)}{\phi_1(l/2)} + \Lambda_0([0,1]) \frac{l}{\phi_1(l)} \frac{\phi_1(l)}{\phi_1(l/2)} \leq \frac{4}{c}[1 + \Lambda_0([0,1]) \frac{l}{\phi_1(l)}] \leq C.$$

Therefore, $A^{[n]}f(l) \leq Cf(l)$ and $(e^{-Ct} f(\#\Pi^*_{|[n]}(t)))_{t \leq \tau}$ is a supermartingale. □

Lemma 1.27 *If* $\sum_{n \geq 1} \frac{1}{\phi(n)} = \infty$ *then* Π *does not come down from infinity.*

Proof. Assume that the M-coalescent comes down from infinity. By Proposition 1.15, we know that $T = 0$ a.s. Let $T_j^{(n)} := \inf\{t > 0; \#\Pi^*_{|[n]}(t) \leq j\}$. We apply to the previous supermartingale, the optional stopping theorem at time $T_j^{(n)} \wedge \tau_n$ and get

$$\mathbb{E}[e^{-cT_j^{(n)} \wedge \tau_n} f(\#\Pi^*_{|[n]}(\tau_n \wedge T_j^{(n)}))] \leq f(n).$$

Passing to the limit with $n \uparrow \infty$: $f(n) \to 0$, $T_j^{(n)} \uparrow T_j$ and $\tau_n \to \tau > 0$ (by Lemma 1.25). The time T_j is strictly positive for some j, then $\tau \wedge T_j > 0$, $\#\Pi^*(\tau \wedge T_j) < \infty$ and thus, $f(\#\Pi^*(\tau \wedge T_j)) > 0$ almost surely. We have

$$\mathbb{E}[e^{-cT_j \wedge \tau}] = 0$$

then $T_j = \infty$ a.s, which is not possible on $T < \infty$.□

It remains to establish that the convergence of the series is necessary for the coming down from infinity. When $\Lambda_0(\{0\}) = 0$, the previous lemma claims that if the M-coalescent comes down from infinity then $\sum_{n \geq 1} \frac{1}{\phi(n)} < \infty$. It suffices to show that

$$\sum_{n \geq 2} \frac{1}{\phi_1(n) + \Lambda_0([0,1])n} < \infty \Longrightarrow \sum_{n \geq 2} \frac{1}{\phi_1(n)} < \infty.$$

The sequence $(\frac{\Psi(n)}{n})_{n \geq 1}$ is increasing and tends to $\int_0^1 x^{-1}\Lambda_1(dx)$ (possibly infinite). From the inequality : $c\frac{\Psi(n)}{n} \leq \frac{\phi_1(n)}{n} \leq \frac{\Psi(n)}{n}$, we get that $\frac{n}{\phi_1(n)}$ is bounded. It follows that

$$\frac{1}{\phi_1(n) + \Lambda_0([0,1])n} = \frac{1}{\phi_1(n)(1 + \Lambda_0([0,1])\frac{n}{\phi_1(n)})} \geq c\frac{1}{\phi_1(n)}$$

for some constant $c > 0$.
We then get the necessary part, and combining the results, Theorem 1.16 is deduced.

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Chapitre 2

Stable continuous-state branching processes with immigration and Beta-Fleming-Viot processes with immigration

This article has been written with Olivier Hénard.

2.1 Introduction

The connections between the Fleming-Viot processes and the continuous-state branching processes have been intensively studied. Shiga established in 1990 that a Fleming-Viot process may be recovered from the *ratio process* associated with a Feller diffusion up to a random time change, see [Shi90]. This result has been generalized in 2005 by Birkner *et al* in [BBC$^+$05] in the setting of Λ-generalized Fleming-Viot processes and continuous-state branching processes (CBs for short). In that paper they proved that the ratio process associated with an α-stable branching process is a time-changed $Beta(2-\alpha,\alpha)$-Fleming-Viot process for $\alpha \in (0,2)$. The main goal of this article is to study such connections when immigration is incorporated in the underlying population. The continuous-state branching processes with immigration (CBIs for short) are a class of time-homogeneous Markov processes with values in $\mathbb{R}_+$. They have been introduced by Kawazu and Watanabe in 1971, see [KW71], as limits of rescaled Galton-Watson processes with immigration. These processes are characterized by two functions Φ and Ψ respectively called the immigration mechanism and the branching mechanism. A new class of measure-valued processes with immigration has been recently set up in [Fou11](Chapter 1). These processes, called M-generalized Fleming-Viot processes with immigration (M-GFVIs for short) are valued in the space of probability measures on $[0,1]$. The notation M stands for a couple of finite measures (Λ_0,Λ_1) encoding respectively the rates of immigration and of reproduction. The genealogies of the M-GFVIs are given by the so-called M-coalescents. These processes are valued in the space of the partitions of $\mathbb{Z}_+$, denoted by $\mathcal{P}_\infty^0$.

In the same manner as Birkner *et al.* in [BBC$^+$05], Perkins in [Per92] and Shiga in [Shi90], we shall establish some relations between continuous-state branching processes with immigration and M-GFVIs. A notion of continuous population with immigration may be defined using a flow of CBIs in the same spirit as Bertoin and Le Gall in [BLG00].

This allows us to compare the two notions of continuous populations provided respectively by the CBIs and by the M-GFVIs. Using calculations of generators, we show in Theorem 2.3 that the following self-similar CBIs admit time-changed M-GFVIs for ratio processes :

- the Feller branching diffusion with branching rate σ^2 and immigration rate β (namely the CBI with $\Phi(q) = \beta q$ and $\Psi(q) = \frac{1}{2}\sigma^2 q^2$) which has for ratio process a time-changed M-Fleming-Viot process with immigration where $M = (\beta\delta_0, \sigma^2\delta_0)$,
- the CBI process with $\Phi(q) = d'\alpha q^{\alpha-1}$ and $\Psi(q) = dq^{\alpha}$ for some $d, d' \geq 0$, $\alpha \in (1,2)$ which has for ratio process a time-changed M-generalized Fleming-Viot process with immigration where $M = (c'Beta(2-\alpha, \alpha-1), cBeta(2-\alpha, \alpha))$, $c' = \frac{\alpha(\alpha-1)}{\Gamma(2-\alpha)}d'$ and $c = \frac{\alpha(\alpha-1)}{\Gamma(2-\alpha)}d$.

We stress that the CBIs may reach 0, see Proposition 2.1, in which case the M-GFVIs involved describe the ratio process up to this hitting time only. When $d = d'$ or $\beta = \sigma^2$, the corresponding CBIs are respectively the α-stable branching process and the Feller branching diffusion *conditioned to be never extinct*. In that case, the M-coalescents are genuine Λ-coalescent viewed on $\mathcal{P}^0_\infty$. We get respectively a $Beta(2-\alpha, \alpha-1)$-coalescent when $\alpha \in (1,2)$ and a Kingman's coalescent for $\alpha = 2$, see Theorem 2.8. This differs from the α-stable branching process *without immigration* (already studied in [BBC$^+$05]) for which the coalescent involved is a $Beta(2-\alpha, \alpha)$-coalescent.

Outline. The paper is organized as follows. In Section 2, we recall the definition of a continuous-state branching process with immigration and of an M-generalized Fleming-Viot process with immigration. We describe briefly how to define from a flow of CBIs a continuous population represented by a measure-valued process. We state in Section 3 the connections between the CBIs and M-GFVIs, mentioned in the Introduction, and study the random time change. Recalling the definition of an M-coalescent, we focus in Section 4 on the genealogy of the M-GFVIs involved. We establish that, when the CBIs correspond with CB-processes conditioned to be never extinct, the M-coalescents involved are actually classical Λ-coalescents. We identify them and, as mentioned, the $Beta(2-\alpha, \alpha-1)$-coalescent arises. In Section 5, we compare the generators of the M-GFVI and CBI processes and prove the main result.

2.2 A continuous population embedded in a flow of CBIs and the M-generalized Fleming-Viot with immigration

2.2.1 Background on continuous state branching processes with immigration

We will focus on critical continuous-state branching processes with immigration characterized by two functions of the variable $q \geq 0$:

$$\Psi(q) = \frac{1}{2}\sigma^2 q^2 + \int_0^\infty (e^{-qu} - 1 + qu)\hat{\nu_1}(du)$$
$$\Phi(q) = \beta q + \int_0^\infty (1 - e^{-qu})\hat{\nu_0}(du)$$

where $\sigma^2, \beta \geq 0$ and $\hat{\nu_0}$, $\hat{\nu_1}$ are two Lévy measures such that $\int_0^\infty (1 \wedge u)\hat{\nu_0}(du) < \infty$ and $\int_0^\infty (u \wedge u^2)\hat{\nu_1}(du) < \infty$. The measure $\hat{\nu_1}$ is the Lévy measure of a spectrally positive Lévy process which characterizes the reproduction. The measure $\hat{\nu_0}$ characterizes the jumps of the subordinator that describes the arrival of immigrants in the population. The non-negative constants σ^2 and β correspond respectively to the continuous reproduction and the continuous immigration. Let $\mathbb{P}_x$ be the law of a CBI $(Y_t, t \geq 0)$ started at x, and denote by $\mathbb{E}_x$ the associated expectation. The law of the Markov process $(Y_t, t \geq 0)$ can then be characterized by the Laplace transform of its marginal as follows : for every $q > 0$ and $x \in \mathbb{R}_+$,

$$\mathbb{E}_x[e^{-qY_t}] = \exp\left(-xv_t(q) - \int_0^t \Phi(v_s(q))ds\right)$$

where v is the unique non-negative solution of $\frac{\partial}{\partial t}v_t(q) = -\Psi(v_t(q))$, $v_0(q) = q$.

The pair (Ψ, Φ) is known as the branching-immigration mechanism. A CBI process $(Y_t, t \geq 0)$ is said to be conservative if for every $t > 0$ and $x \in [0, \infty[, \mathbb{P}_x[Y_t < \infty] = 1$. A result of Kawazu and Watanabe [KW71] states that $(Y_t, t \geq 0)$ is conservative if and only if for every $\epsilon > 0$

$$\int_0^\epsilon \frac{1}{|\Psi(q)|}dq = \infty.$$

Moreover, we shall say that the CBI process is *critical* when $\Psi'(0) = 0$: in that case, the CBI process is necessarily conservative. We follow the seminal idea of Bertoin and Le Gall in [BLG00] to define a genuine continuous population model with immigration on $[0, 1]$ associated with a CBI. Emphasizing the rôle of the initial value, we denote by $(Y_t(x), t \geq 0)$ a CBI started at $x \in \mathbb{R}_+$. The branching property ensures that $(Y_t(x+y), t \geq 0) \overset{law}{=} (Y_t(x) + X_t(y), t \geq 0)$ where $(X_t(y), t \geq 0)$ is a CBI$(\Psi, 0)$ starting from y (that is a CB-process without immigration and with branching mechanism Ψ) independent of $(Y_t(x), t \geq 0)$. The Kolmogorov's extension theorem allows one to construct a flow $(Y_t(x), t \geq 0, x \geq 0)$ such that for every $y \geq 0$, $(Y_t(x + y) - Y_t(x), t \geq 0)$ has the same law as $(X_t(y), t \geq 0)$ a CB-process started from y. We denote by $(M_t, t \geq 0)$ the Stieltjes-measure associated with the increasing process $x \in [0, 1] \mapsto Y_t(x)$. Namely, define

$$M_t(]x, y]) := Y_t(y) - Y_t(x), \quad 0 \leq x \leq y \leq 1.$$
$$M_t(\{0\}) := Y_t(0).$$

The process $(Y_t(1), t \geq 0)$ is assumed to be conservative, therefore the process $(M_t, t \geq 0)$ is valued in the space $\mathcal{M}_f$ of finite measures on $[0, 1]$. We stress that this space is locally compact, separable and metrizable. Therefore we may consider its one-point compactification (see [Li11] page 7). By a slight abuse of notation, we denote by $(Y_t, t \geq 0)$ the process $(Y_t(1), t \geq 0)$. The framework of measure-valued processes allows us to consider an infinitely many types model. Namely each individual has initially its own type (which lies in $[0, 1]$) and transmits it to its progeny. People issued from the immigration have a *distinguished* type fixed at 0. Since the types do not evolve in time, they allow us to track the ancestors at time 0. This model can be viewed as a superprocess without spatial motion (or without mutation in population genetics vocable).
Let $\mathcal{C}$ be the class of functions on $\mathcal{M}_f$ of the form

$$F(\eta) := G\left(\langle f_1, \eta\rangle, ..., \langle f_n, \eta\rangle\right),$$

where $\langle f, \eta\rangle := \int_{[0,1]} f(x)\eta(dx)$, $G \in C^2(\mathbb{R}^n)$ and $f_1, ..., f_n$ are bounded measurable functions on $[0, 1]$. Section 9.3 of Li's book [Li11] (see Theorem 9.18 p. 218) ensures that the following operator acting on the space $\mathcal{M}_f$ is an extended generator of $(M_t, t \geq 0)$. For any $\eta \in \mathcal{M}_f$,

$$\mathcal{L}F(\eta) := \sigma^2/2 \int_0^1 \int_0^1 \eta(da)\delta_a(db)F''(\eta; a, b) \tag{2.1}$$
$$+ \beta F'(\eta; 0) \tag{2.2}$$
$$+ \int_0^1 \eta(da) \int_0^\infty \hat{\nu_1}(dh)[F(\eta + h\delta_a) - F(\eta) - hF'(\eta, a)] \tag{2.3}$$
$$+ \int_0^\infty \hat{\nu_0}(dh)[F(\eta + h\delta_0) - F(\eta)] \tag{2.4}$$

where $F'(\eta; a) := \lim_{\epsilon\to 0} \frac{1}{\epsilon}[F(\eta + \epsilon\delta_a) - F(\eta)]$ is the Gateaux derivative of F at η in direction δ_a, and $F''(\eta; a, b) := G'(\eta; b)$ with $G(\eta) = F'(\eta; a)$. The terms (1) and (3) correspond to the reproduction, see for instance Section 6.1 p. 106 of Dawson [Daw93]. The terms (2) and (4) correspond to the immigration. We stress that in our model the immigration is concentrated on 0, contrary to other works which consider infinitely many types for the immigrants. For the interested reader, the operator $\mathcal{L}$ corresponds with that given in equation (9.25) of Section 9 of Li [Li11] by setting $H(d\mu) = \int_0^\infty \hat{\nu}_0(dh)\delta_{h\delta_0}(d\mu)$ and $\eta = \beta\delta_0$.

For all $\eta \in \mathcal{M}_f$, we denote by $|\eta|$ the total mass $|\eta| := \eta([0, 1])$. If $(M_t, t \geq 0)$ is a Markov process with the above operator for generator, the process $(|M_t|, t \geq 0)$ is by construction a CBI. This is also plain from the form of the generator $\mathcal{L}$: let ψ be a twice differentiable function on $\mathbb{R}_+$ and define $F : \eta \mapsto \psi(|\eta|)$, we find $\mathcal{L}F(\eta) = zG_B\psi(z)+G_I\psi(z)$ for $z = |\eta|$, where

$$G_B\psi(z) = \frac{\sigma^2}{2}\psi''(z) + \int_0^\infty [\psi(z + h) - \psi(z) - h\psi'(z)]\hat{\nu_1}(dh) \tag{2.5}$$
$$G_I\psi(z) = \beta\psi'(z) + \int_0^\infty [\psi(z + h) - \psi(z)]\hat{\nu_0}(dh). \tag{2.6}$$

2.2.2 Background on M-generalized Fleming-Viot processes with immigration

We denote by $\mathcal{M}_1$ the space of probability measures on $[0, 1]$. Let c_0, c_1 be two nonnegative real numbers and ν_0, ν_1 be two measures on $[0, 1]$ such that $\int_0^1 x\nu_0(dx) < \infty$ and $\int_0^1 x^2\nu_1(dx) < \infty$. Following the notation of [Fou11], we define the couple of finite measures $M = (\Lambda_0, \Lambda_1)$ such that

$$\Lambda_0(dx) = c_0\delta_0(dx) + x\nu_0(dx),\ \Lambda_1(dx) = c_1\delta_0(dx) + x^2\nu_1(dx).$$

The M-generalized Fleming-Viot process with immigration describes a population with *constant size* which evolves by resampling. Let $(\rho_t, t \geq 0)$ be an M-generalized Fleming-Viot process with immigration. The evolution of this process is a superposition of a continuous evolution, and a discontinuous one. The continuous evolution can be described as follows : every couple of individuals is sampled at constant rate c_1, in which case one of the two individuals gives its type to the other : this is a reproduction event. Furthermore, any individual is picked at constant rate c_0, and its type replaced by the distinguished type 0 (the immigrant type) : this is an immigration event. The discontinuous evolution is prescribed by two independent Poisson point measures N_0 and N_1 on $\mathbb{R}_+ \times [0, 1]$ with respective intensity $dt \otimes \nu_0(dx)$ and $dt \otimes \nu_1(dx)$. More precisely, if (t, x) is an atom of $N_0 + N_1$ then t is a jump time of the process $(\rho_t, t \geq 0)$ and the conditional law of ρ_t given ρ_{t-} is :

– $(1-x)\rho_{t-} + x\delta_U$, if (t,x) is an atom of N_1, where U is distributed according to ρ_{t-}

– $(1-x)\rho_{t-} + x\delta_0$, if (t,x) is an atom of N_0.

If (t,x) is an atom of N_1, an individual is picked at random in the population at generation $t-$ and generates a proportion x of the population at time t : this is a reproduction event, as for the genuine generalized Fleming-Viot process (see [BLG03] p278). If (t,x) is an atom of N_0, the individual 0 at time $t-$ generates a proportion x of the population at time t : this is an immigration event. In both cases, the population at time $t-$ is reduced by a factor $1-x$ so that, at time t, the total size is still 1. The genealogy of this population (which is identified as a probability measure on $[0,1]$) is given by an M-coalescent (see Section 2.4 below). This description is purely heuristic (we stress for instance that the atoms of $N_0 + N_1$ may form an infinite dense set), to make a rigorous construction of such processes, we refer to the Section 5.2 of [Fou11] (or alternatively Section 3.2 of [Fou12]). For any $p \in \mathbb{N}$ and any continuous function f on $[0,1]^p$, we denote by G_f the map

$$\rho \in \mathcal{M}_1 \mapsto \langle f, \rho^{\otimes p}\rangle := \int_{[0,1]^p} f(x)\rho^{\otimes p}(dx) = \int_{[0,1]^p} f(x_1, ..., x_p)\rho(dx_1)...\rho(dx_p).$$

Let $(\mathcal{F}, \mathcal{D})$ denote the generator of $(\rho_t, t \geq 0)$ and its domain. The vector space generated by the functionals of the type G_f forms a core of $(\mathcal{F}, \mathcal{D})$ and we have (see Lemma 5.2 in [Fou11]) :

$$\mathcal{F}G_f(\rho) = c_1 \sum_{1\leq i<j\leq p} \int_{[0,1]^p} [f(\mathrm{x}^{i,j}) - f(\mathrm{x})]\rho^{\otimes p}(d\mathrm{x}) \quad (1')$$

$$+ c_0 \sum_{1\leq j\leq p} \int_{[0,1]^p} [f(\mathrm{x}^{0,j}) - f(\mathrm{x})]\rho^{\otimes p}(d\mathrm{x}) \quad (2')$$

$$+ \int_0^1 \nu_1(dr) \int \rho(da)[G_f((1-r)\rho + r\delta_a) - G_f(\rho)] \quad (3')$$

$$+ \int_0^1 \nu_0(dr)[G_f((1-r)\rho + r\delta_0) - G_f(\rho)]. \quad (4')$$

where x denotes the vector $(x_1, ..., x_p)$ and

– the vector $\mathrm{x}^{0,j}$ is defined by $\mathrm{x}_k^{0,j} = x_k$, for all $k \neq j$ and $\mathrm{x}_j^{0,j} = 0$,

– the vector $\mathrm{x}^{i,j}$ is defined by $\mathrm{x}_k^{i,j} = x_k$, for all $k \neq j$ and $\mathrm{x}_j^{i,j} = x_i$.

2.3 Relations between CBIs and M-GFVIs

2.3.1 Forward results

The expressions of the generators of $(M_t, t \geq 0)$ and $(\rho_t, t \geq 0)$ lead us to specify the connections between CBIs and GFVIs. We add a cemetery point Δ to the space $\mathcal{M}_1$ and define $(R_t, t \geq 0) := (\frac{M_t}{|M_t|}, t \geq 0)$, the ratio process with lifetime $\tau := \inf\{t \geq 0; |M_t| = 0\}$. By convention, for all $t \geq \tau$, we set $R_t = \Delta$. As mentioned in the Introduction, we shall focus our study on the two following critical CBIs :

(i) $(Y_t, t \geq 0)$ is CBI with parameters $\sigma^2, \beta \geq 0$ and $\hat{\nu}_0 = \hat{\nu}_1 = 0$, so that $\Psi(q) = \frac{\sigma^2}{2}q^2$ and $\Phi(q) = \beta q$.

(ii) $(Y_t, t \geq 0)$ is a CBI with $\sigma^2 = \beta = 0$, $\hat{\nu_0}(dh) = c'h^{-\alpha}1_{h>0}dh$ and $\hat{\nu_1}(dh) = ch^{-1-\alpha}1_{h>0}dh$ for $1 < \alpha < 2$, so that $\Psi(q) = dq^\alpha$ and $\Phi(q) = d'\alpha q^{\alpha-1}$ with $d' = \frac{\Gamma(2-\alpha)}{\alpha(\alpha-1)}c'$ and $d = \frac{\Gamma(2-\alpha)}{\alpha(\alpha-1)}c$

Notice that the CBI in (i) may be seen as a limit case of the CBIs in (ii) for $\alpha = 2$. We first establish in the following proposition a dichotomy for the finiteness of the lifetime, depending on the ratio immigration over reproduction.

Proposition 2.1 *Recall the notation* $\tau = \inf\{t \geq 0, Y_t = 0\}$.
- *If* $\frac{\beta}{\sigma^2} \geq \frac{1}{2}$ *in case (i) or* $\frac{c'}{c} \geq \frac{\alpha-1}{\alpha}$ *in case (ii), then* $\mathbb{P}[\tau = \infty] = 1$.
- *If* $\frac{\beta}{\sigma^2} < \frac{1}{2}$ *in case (i) or* $\frac{c'}{c} < \frac{\alpha-1}{\alpha}$ *in case (ii), then* $\mathbb{P}[\tau < \infty] = 1$.

We then deal with the random change of time. In the case of a CB-process (that is a CBI process without immigration), Birkner *et al.* used the Lamperti representation and worked on the embedded stable spectrally positive Lévy process. We shall work directly on the CBI process instead. For $0 \leq t \leq \tau$, we define :

$$C(t) = \int_0^t Y_s^{1-\alpha} ds,$$

in case (ii) and set $\alpha = 2$ in case (i).

Proposition 2.2 *In both cases (i) and (ii), we have :*

$$\mathbb{P}\left(C(\tau) = \infty\right) = 1.$$

In other words, the additive functional C *maps* $[0, \tau[$ *to* $[0, \infty[$.

By convention, if τ is almost surely finite we set $C(t) = C(\tau) = \infty$ for all $t \geq \tau$. Denote by C^{-1} the right continuous inverse of the functional C. This maps $[0, \infty[$ to $[0, \tau[$, a.s. We stress that in most cases, $(R_t, t \geq 0)$ is not a Markov process. Nevertheless, in some cases, through a change of time, the process $(R_t, t \geq 0)$ may be changed into a Markov process. This shall be stated in the following Theorem where the functional C is central. For every $x, y > 0$, denote by $Beta(x, y)(dr)$ the finite measure with density

$$r^{x-1}(1-r)^{y-1}1_{(0,1)}(r)dr,$$

and recall that its total mass is given by the Beta function $B(x, y)$.

Theorem 2.3 *Let* $(M_t, t \geq 0)$ *be the measure-valued process associated to a process* $(Y_t(x), x \in [0,1], t \geq 0)$.

- *In case (i), the process* $(R_{C^{-1}(t)})_{t \geq 0}$ *is a* M*-Fleming-Viot process with immigration with*
$$\Lambda_0(dr) = \beta\delta_0(dr) \text{ and } \Lambda_1(dr) = \sigma^2\delta_0(dr).$$
- *In case (ii), the process* $(R_{C^{-1}(t)})_{t \geq 0}$ *is a* M*-generalized Fleming-Viot process with immigration with*

$$\Lambda_0(dr) = c'Beta(2-\alpha, \alpha-1)(dr) \text{ and } \Lambda_1(dr) = cBeta(2-\alpha, \alpha)(dr).$$

The proof requires rather technical arguments on the generators and is given in Section 2.5.

Remark 1 – *The CBIs in the statement of Theorem 2.3 with $\sigma^2 = \beta$ in case* (i) *or* $c = c'$ *in case* (ii)*, are also CBs conditioned on non extinction and are studied further in Section 2.4.*

– *Contrary to the case without immigration, see Theorem 1.1 in [BBC+05], we have to restrict ourselves to* $\alpha \in (1, 2]$.

So far, we state that the ratio process $(R_t, t \geq 0)$ associated to $(M_t, t \geq 0)$, once time changed by C^{-1}, is a M-GFVI process. Conversely, starting from a M-GFVI process, we could wonder how to recover the measure-valued CBI process $(M_t, t \geq 0)$. This lead us to investigate the relation between the time changed ratio process $(R_{C^{-1}(t)}, t \geq 0)$ and the process $(Y_t, t \geq 0)$.

Proposition 2.4 *In case* (i) *of Theorem 2.3, the additive functional* $(C(t), t \geq 0)$ *and* $(R_{C^{-1}(t)}, 0 \leq t < \tau)$ *are independent.*

This proves that in case (i) we need additional randomness to reconstruct M from the M-GFVI process. On the contrary, in case (ii), the process $(Y_t, t \geq 0)$ is clearly not independent of the ratio process $(R_t, t \geq 0)$, since both processes jump at the same time. The proof of Propositions 2.1, 2.2 are given in the next Subsection. Some rather technical arguments are needed to prove Proposition 2.4. We postpone its proof to the end of Section 2.5.

2.3.2 Proofs of Propositions 2.1, 2.2

Proof of Proposition 2.1. Let $(X_t(x), t \geq 0)$ denote an α-stable branching process started at x (with $\alpha \in (1, 2]$). Denote ζ its absorption time, $\zeta := \inf\{t \geq 0; X_t(x) = 0\}$. The following construction of the process $(Y_t(0), t \geq 0)$ may be deduced from the expression of the Laplace transform of the CBI process. We shall need the canonical measure $\mathbb{N}$ which is a sigma-finite measure on càdlàg paths and represents informally the "law" of the population generated by one single individual in a CB(Ψ), see Chu and Ren [CR11]. We write :

$$(Y_t(0), t \geq 0) = \left(\sum_{i \in \mathcal{I}} X^i_{(t-t_i)_+}, t \geq 0 \right) \tag{2.7}$$

with $\sum_i \delta_{(t_i, X^i)}$ a Poisson random measure on $\mathbb{R}_+ \times \mathcal{D}(\mathbb{R}_+, \mathbb{R}_+)$ with intensity $dt \otimes \mu$, where $\mathcal{D}(\mathbb{R}_+, \mathbb{R}_+)$ denotes the space of càdlàg functions, and μ is defined as follows :

– in case (ii), $\mu(dX) = \int \hat{\nu_0}(dx) \mathbb{P}_x(dX)$, where $\mathbb{P}_x$ is the law of a CB(Ψ) with $\Psi(q) = dq^\alpha$. Formula (2.7) may be understood as follows : at the jump times t_i of a pure jump stable subordinator with Lévy measure $\hat{\nu_0}$, a new arrival of immigrants, of size X^i_0, occurs in the population. Each of these "packs", labelled by $i \in \mathcal{I}$, generates its own descendance $(X^i_t, t \geq 0)$, which is a CB(Ψ) process.

– in case (i), $\mu(dX) = \beta\ \mathbb{N}(dX)$, where $\mathbb{N}$ is the canonical measure associated to the CB(Ψ) with $\Psi(q) = \frac{\sigma^2}{2} q^2$. The canonical measure may be thought of as the "law" of the population generated by one single individual. The link with case (ii) is the following : the pure jump subordinator degenerates into a continuous subordinator equal to $(t \mapsto \beta t)$. The immigrants no more arrive by packs, but appear continuously.

Actually, the canonical measure $\mathbb{N}$ is defined in both cases (i) and (ii), and we may always write $\mu(dX) = \Phi(\mathbb{N}(dX))$. The process $(Y_t(0), t \geq 0)$ is a CBI(Ψ, Φ) started at 0. We call

$\mathcal{R}$ the set of zeros of $(Y_t(0), t > 0)$:

$$\mathcal{R} := \{t > 0; Y_t(0) = 0\}.$$

Denote $\zeta_i = \inf\{t > 0, X_t^i = 0\}$ the lifetime of the branching process X^i. The intervals $]t_i, t_i + \zeta_i[$ and $[t_i, t_i + \zeta_i[$ represent respectively the time where X^i is alive in case (i) and in case (ii) (in this case, we have $X_{t_i}^i > 0$.) Therefore, if we define $\tilde{\mathcal{R}}$ as the set of the positive real numbers left uncovered by the random intervals $]t_i, t_i + \zeta_i[$, that is :

$$\tilde{\mathcal{R}} := \mathbb{R}_+^\star \setminus \bigcup_{i \in \mathcal{I}}]t_i, t_i + \zeta_i[.$$

we have $\mathcal{R} \subset \tilde{\mathcal{R}}$ with equality in case (i) only.

The lengths ζ_i have law $\mu(\zeta \in dt)$ thanks to the Poisson construction of $Y(0)$. We now distinguish the two cases :

– Feller case : this corresponds to $\alpha = 2$. We have $\Psi(q) := \frac{\sigma^2}{2} q$ and $\Phi(q) := \beta q$, and thus
$$\mu[\zeta > t] = \beta\, \mathbb{N}[\zeta > t] = \frac{2\beta}{\sigma^2} \frac{1}{t}$$
see Li [Li11] p. 62. Using Example 1 p. 180 of Fitzsimmons et al. [FFS85], we deduce that
$$\tilde{\mathcal{R}} = \emptyset \ \text{ a.s. if and only if } \ \frac{2\beta}{\sigma^2} \geq 1. \tag{2.8}$$

– Stable case : this corresponds to $\alpha \in (1, 2)$. Recall $\Psi(q) := dq^\alpha, \Phi(q) := d'\alpha q^{\alpha-1}$. In that case, we have,
$$\mathbb{N}(\zeta > t) = d^{-\frac{1}{\alpha-1}} [(\alpha-1)t]^{-\frac{1}{\alpha-1}}.$$
Thus, $\mu[\zeta > t] = \Phi(\mathbb{N}(\zeta > t)) = \frac{\alpha}{\alpha-1} \frac{d'}{d} \frac{1}{t}$. Recall that $\frac{d'}{d} = \frac{c'}{c}$. Therefore, using reference [FFS85], we deduce that
$$\tilde{\mathcal{R}} = \emptyset \ \text{ a.s. if and only if } \ \frac{c'}{c} \geq \frac{\alpha - 1}{\alpha}. \tag{2.9}$$

This allows us to establish the first point of Proposition 2.1 : we get $\mathcal{R} \subset \tilde{\mathcal{R}} = \emptyset$, and the inequality $Y_t(1) \geq Y_t(0)$ for all t ensures that $\tau = \infty$.

We deal now with the second point of Proposition 2.1. Assume that $\frac{c'}{c} < \frac{\alpha-1}{\alpha}$ or $\frac{\beta}{\sigma^2} < \frac{1}{2}$. By assertions (2.8) and (2.9), we already know that $\tilde{\mathcal{R}} \neq \emptyset$. However, what we really need is that $\tilde{\mathcal{R}}$ is a.s. not bounded. To that aim, observe that, in both cases (i) and (ii),

$$\mu[\zeta > s] = \Phi(\mathbb{N}(\zeta > s)) = \frac{\kappa}{s}$$

with $\kappa = \frac{\alpha}{\alpha-1} \frac{d'}{d} = \frac{\alpha}{\alpha-1} \frac{c'}{c} < 1$ if $1 < \alpha < 2$ and $\kappa = \frac{2\beta}{\sigma^2} < 1$ if $\alpha = 2$. Thus $\int_1^u \mu[\zeta > s] ds = \kappa \ln(u)$ and we obtain
$$\exp\left(-\int_1^u \mu[\zeta > s] ds\right) = \left(\frac{1}{u}\right)^\kappa.$$

Therefore, since $\kappa < 1$,

$$\int_1^\infty \exp\left(-\int_1^u \mu[\zeta > s] ds\right) du = \infty,$$

which implies thanks to Corollary 4 (Equation 17 p 183) of [FFS85] that $\tilde{\mathcal{R}}$ is a.s. not bounded.

Since $\mathcal{R} = \tilde{\mathcal{R}}$ in case (i), the set $\mathcal{R}$ is a.s. not bounded in that case. Now, we prove that $\mathcal{R}$ is a.s. not bounded in case (ii). The set $\tilde{\mathcal{R}}$ is almost surely not empty and not bounded. Moreover this is a perfect set (Corollary 1 of [FFS85]). Since there are only countable points $(t_i, i \in \mathcal{I})$, the set $\tilde{\mathcal{R}} = \mathcal{R} \setminus \bigcup_{i\in I}\{t_i\}$ is also uncountable and not bounded.

Last, recall from Subsection 2.2.1 that we may write $Y_t(1) = Y_t(0) + X_t(1)$ for all $t \geq 0$ with $(X_t(1), t \geq 0)$ a CB-process independent of $(Y_t(0), t \geq 0)$. Let $\xi := \inf\{t \geq 0, X_t(1) = 0\}$ be the extinction time of $(X_t(1), t \geq 0)$. Since $\mathcal{R}$ is a.s. not bounded in both cases (i) and (ii), $\mathcal{R} \cap (\xi, \infty) \neq \emptyset$, and $\tau < \infty$ almost surely. □

Proof of Proposition 2.2. Recall that $Y_t(x)$ is the value of the CBI started at x at time t. We will denote by $\tau^x(0) := \inf\{t > 0, Y_t(x) = 0\}$. With this notation, $\tau^1(0) = \tau$ introduced in Section 2.3.1. In both cases (i) and (ii), the processes are self-similar, see Kyprianou and Pardo [KP08] or Lemma 4.8 in [Pat09]. Namely, we have

$$(xY_{x^{1-\alpha}t}(1), t \geq 0) \overset{law}{=} (Y_t(x), t \geq 0)\,,$$

where we take $\alpha = 2$ in case (i). Performing the change of variable $s = x^{1-\alpha}t$, we obtain

$$\int_0^{\tau^x(0)} dt\ Y_t(x)^{1-\alpha} \overset{law}{=} \int_0^{\tau^1(0)} ds\ Y_s(1)^{1-\alpha}. \tag{2.10}$$

According to Proposition 2.1, depending on the values of the parameters :

- Either $\mathbb{P}(\tau^x(0) < \infty) = 1$ for every x. Let $x > 1$. Denote $\tau^x(1) = \inf\{t > 0, Y_t(x) \leq 1\}$. We have $\mathbb{P}(\tau^x(1) < \infty) = 1$. We have :

$$\int_0^{\tau^x(0)} dt\ Y_t(x)^{1-\alpha} = \int_0^{\tau^x(1)} dt\ Y_t(x)^{1-\alpha} + \int_{\tau^x(1)}^{\tau^x(0)} dt\ Y_t(x)^{1-\alpha}$$

  By the strong Markov property applied at the stopping time $\tau^x(1)$, since Y has no negative jumps :

$$\int_{\tau^x(1)}^{\tau^x(0)} dt\ Y_t(x)^{1-\alpha} \overset{law}{=} \int_0^{\tau^1(0)} dt\ \tilde{Y}_t(1)^{1-\alpha},$$

  with $(\tilde{Y}_t(1), t \geq 0)$ an independent copy started from 1. Since

$$\int_0^{\tau^x(1)} dt\ Y_t(x)^{1-\alpha} > 0,\ a.s.,$$

  the equality (2.10) is impossible unless both sides of the equality are infinite almost surely. We thus get that $C(\tau) = \infty$ almost surely in that case.

- Either $\mathbb{P}(\tau^x(0) = \infty) = 1$ for every x, on which case we may rewrite (2.10) as follows :

$$\int_0^{\infty} dt\ Y_t(x)^{1-\alpha} \overset{law}{=} \int_0^{\infty} ds\ Y_s(1)^{1-\alpha}.$$

  Since, for $x > 1$, the difference $(Y_t(x) - Y_t(1), t \geq 0)$ is an α-stable CB-process started at $x - 1 > 0$, we deduce that $C(\tau) = \infty$ almost surely again.

This proves the statement. □

Remark 2 *The situation is quite different when the CBI process starts at* 0*, in which case the time change also diverges in the neighbourhood of* 0*. The same change of variables as in (2.10) yields, for all* $0 < x < k$*,*

$$\int_0^{\iota^x(k)} dt\ Y_t(x)^{1-\alpha} \overset{law}{=} \int_0^{\iota^1(k/x)} dt\ Y_t(1)^{1-\alpha},$$

with $\iota^x(k) = \inf\{t > 0, Y_t(x) \geq k\} \in [0,\infty]$*. Letting* x *tend to* 0*, we get* $\iota^1(k/x) \longrightarrow \infty$ *and the right hand side diverges to infinity. Thus, the left hand side also diverges, which implies that :*

$$\mathbb{P}\left(\int_0^{\iota^0(k)} dt\ Y_t(0)^{1-\alpha} = \infty\right) = 1.$$

2.4 Genealogy of the Beta-Fleming-Viot processes with immigration

To describe the genealogy associated with stable CBs, Bertoin and Le Gall [BLG06] and Birkner et al. [BBC+05] used partition-valued processes called Beta-coalescents. These processes form a subclass of Λ-coalescents, introduced independently by Pitman and Sagitov in 1999. A Λ-coalescent is an exchangeable process in the sense that its law is invariant under the action of any permutation. In words, there is no distinction between the individuals. Although these processes arise as models of genealogy for a wide range of stochastic populations, they are not in general adapted to describe the genealogy of a population with immigration. Recently, a larger class of processes called M-coalescents has been defined in [Fou11] (see Section 5). These processes are precisely those describing the genealogy of M-GFVIs.

Remark 3 *We mention that the use of the lookdown construction in Birkner et al. [BBC+05] may be easily adapted to our framework and yields a genealogy for any conservative CBI. Moreover, other genealogies, based on continuous trees, have been investigated by Lambert [Lam02] and Duquesne [Duq09].*

2.4.1 Background on M-coalescents

Before focusing on the M-coalescents involved in the context of Theorem 2.3, we recall their general definition and the duality with the M-GFVIs. Contrary to the Λ-coalescents, the M-coalescents are only invariant by permutations letting 0 fixed. The individual 0 represents the immigrant lineage and is distinguished from the others. We denote by $\mathcal{P}_\infty^0$ the space of partitions of $\mathbb{Z}_+ := \{0\}\bigcup\mathbb{N}$. Let $\pi \in \mathcal{P}_\infty^0$. By convention, we identify π with the sequence $(\pi_0, \pi_1, ...)$ of the blocks of π enumerated in increasing order of their smallest element : for every $i \leq j$, $\min \pi_i \leq \min \pi_j$. Let $[n]$ denote the set $\{0, ..., n\}$ and $\mathcal{P}_n^0$ the space of partitions of $[n]$. The partition of $[n]$ into singletons is denoted by $0_{[n]}$. As in Section 2.2, the notation M stands for a pair of finite measures (Λ_0, Λ_1) such that :

$$\Lambda_0(dx) = c_0\delta_0(dx) + x\nu_0(dx),\ \Lambda_1(dx) = c_1\delta_0(dx) + x^2\nu_1(dx),$$

where c_0, c_1 are two non-negative real numbers and ν_0, ν_1 are two measures on $[0,1]$ subject to the same conditions as in Section 2.2.2. Let N_0 and N_1 be two Poisson point measures with intensity respectively $dt\otimes\nu_0$ and $dt\otimes\nu_1$. An M-coalescent is a Feller process $(\Pi(t), t \geq 0)$ valued in $\mathcal{P}_\infty^0$ with the following dynamics.

– At an atom (t, x) of N_1, flip a coin with probability of "heads" x for each block not containing 0. All blocks flipping "heads" are merged immediately in one block. At time t, a proportion x share a common parent in the population.

– At an atom (t, x) of N_0, flip a coin with probability of "heads" x for each block not containing 0. All blocks flipping "heads" coagulate immediately with the distinguished block. At time t, a proportion x of the population is children of immigrant.

In order to take into account the parameters c_0 and c_1, imagine that at constant rate c_1, two blocks (not containing 0) merge *continuously* in time, and at constant rate c_0, one block (not containing 0) merged with the distinguished one. We refer to Section 4.2 of [Fou11] for a rigorous definition. Let $\pi \in \mathcal{P}_n^0$. The jump rate of an M-coalescent from $0_{[n]}$ to π, denoted by q_π, is given as follows :

– If π has one block not containing 0 with k elements and $2 \le k \le n$, then

$$q_\pi = \lambda_{n,k} := \int_0^1 x^{k-2}(1-x)^{n-k}\Lambda_1(dx).$$

– If the distinguished block of π has $k+1$ elements (counting 0) and $1 \le k \le n$ then

$$q_\pi = r_{n,k} := \int_0^1 x^{k-1}(1-x)^{n-k}\Lambda_0(dx).$$

The next duality property is a key result and links the M-GFVIs to the M-coalescents. For any π in $\mathcal{P}_\infty^0$, define

$$\alpha_\pi : k \mapsto \text{the index of the block of } \pi \text{ containing } k.$$

We have the duality relation (see Lemma 4 in [Fou12]) : for any $p \ge 1$ and $f \in C([0,1]^p)$,

$$\mathbb{E}\left[\int_{[0,1]^{p+1}} f(x_{\alpha_{\Pi(t)}(1)}, ..., x_{\alpha_{\Pi(t)}(p)})\delta_0(dx_0)dx_1...dx_p\right] = \mathbb{E}\left[\int_{[0,1]^p} f(x_1, ..., x_p)\rho_t(dx_1)...\rho_t(dx_p)\right],$$

where $(\rho_t, t \ge 0)$ is a M-GFVI started from the Lebesgue measure on $[0,1]$. We establish a useful lemma relating genuine Λ-coalescents and M-coalescents. Consider a Λ-coalescent taking values in the set $\mathcal{P}_\infty^0$; this differs from the usual convention, according to which they are valued in the set $\mathcal{P}_\infty$ of the partitions of $\mathbb{N}$ (see Chapters 1 and 3 of [Ber09] for a complete introduction to these processes). In that framework, Λ-coalescents appear as a subclass of M-coalescents and the integer 0 may be viewed as a typical individual. The proof is postponed in Section 2.4.3.

Lemma 2.5 *A M-coalescent, with $M = (\Lambda_0, \Lambda_1)$ is also a Λ-coalescent on $\mathcal{P}_\infty^0$ if and only if*

$$(1-x)\Lambda_0(dx) = \Lambda_1(dx).$$

In that case $\Lambda = \Lambda_0$.

2.4.2 The $Beta(2-\alpha, \alpha-1)$-coalescent

The aim of this Section is to show how a $Beta(2-\alpha, \alpha-1)$-coalescent is embedded in the genealogy of an α-stable CB-process conditioned to be never extinct. Along the way, we also derive the fixed time genealogy of the Feller CBI.

We first state the following straightforward Corollary of Theorem 2.3, which gives the genealogy of the ratio process at the random time $C^{-1}(t)$:

Corollary 2.6 *Let $(R_t, t \geq 0)$ be the ratio process of a CBI in case (i) or (ii). We have for all $t \geq 0$:*

$$\mathbb{E}\left[\int_{[0,1]^{p+1}} f(x_{\alpha_{\Pi(t)}(1)}, ..., x_{\alpha_{\Pi(t)}(p)})\delta_0(dx_0)dx_1...dx_p\right] = \mathbb{E}\left[\int_{[0,1]^p} f(x_1, ..., x_p)R_{C^{-1}(t)}(dx_1)...R_{C^{-1}(t)}(dx_p)\right],$$

where :

- *In case (i), $(\Pi(t), t \geq 0)$ is a M-coalescent with $M = (\beta\delta_0, \sigma^2\delta_0)$,*
- *In case (ii), $(\Pi(t), t \geq 0)$ is a M-coalescent with $M = (c'Beta(2-\alpha, \alpha-1), cBeta(2-\alpha, \alpha))$.*

In general, we cannot set the random quantity $C(t)$ instead of t in the equation of Corollary 2.6. Nevertheless, using the independence property proved in Proposition 2.4, we get the following Corollary, whose proof may be found in Section 2.4.3..

Corollary 2.7 *In case (i), assume $\frac{\beta}{\sigma^2} \geq \frac{1}{2}$, then for all $t \geq 0$,*

$$\mathbb{E}\left[\int_{[0,1]^{p+1}} f(x_{\alpha_{\Pi(C(t))}(1)}, ..., x_{\alpha_{\Pi(C(t))}(p)})\delta_0(dx_0)dx_1...dx_p\right] = \mathbb{E}\left[\int_{[0,1]^p} f(x_1, ..., x_p)R_t(dx_1)...R_t(dx_p)\right],$$

where $(\Pi(t), t \geq 0)$ is a M-coalescent with $M = (\beta\delta_0, \sigma^2\delta_0)$, $(Y_t, t \geq 0)$ is a CBI in case (i) independent of $(\Pi(t), t \geq 0)$ and $(C(t), t \geq 0) = \left(\int_0^t \frac{1}{Y_s}ds, t \geq 0\right)$.

We stress on a fundamental difference between Corollaries 2.6 and 2.7. Whereas the first gives the genealogy of the ratio process R at the random time $C^{-1}(t)$, the second gives the genealogy of the ratio process R at a fixed time t. Notice that we impose the additional assumption that $\frac{\beta}{\sigma^2} \geq \frac{1}{2}$ in Corollary 2.7 for ensuring that the lifetime is infinite. Therefore, $R_t \neq \Delta$ for all $t \geq 0$, and we may consider its genealogy.

We easily check that the M-coalescents for which $M = (\sigma^2\delta_0, \sigma^2\delta_0)$ and $M = (cBeta(2-\alpha, \alpha-1), cBeta(2-\alpha, \alpha))$ fulfill the conditions of Lemma 2.5. Recall from Section 2.3.1 the definitions of the CBIs in case (i) and (ii) .

Theorem 2.8 *(i) If the process $(Y_t, t \geq 0)$ is a CBI such that $\sigma^2 = \beta > 0$, $\hat{\nu}_1 = \hat{\nu}_0 = 0$, then the process $(\Pi(t/\sigma^2), t \geq 0)$ defined in Corollary 2.6 is a Kingman's coalescent valued in $\mathcal{P}_\infty^0$.*

(ii) If the process $(Y_t, t \geq 0)$ is a CBI such that $\sigma^2 = \beta = 0$ and $\hat{\nu}_0(dh) = ch^{-\alpha}dh$, $\hat{\nu}_1(dh) = ch^{-\alpha-1}dh$ for some constant $c > 0$ then the process $(\Pi(t/c), t \geq 0)$ defined in Corollary 2.6 is a $Beta(2-\alpha, \alpha-1)$-coalescent valued in $\mathcal{P}_\infty^0$.

In both cases, the process $(Y_t, t \geq 0)$ involved in that Theorem may be interpreted as a CB-process $(X_t, t \geq 0)$ without immigration ($\beta = 0$ or $c' = 0$) conditioned on non-extinction, see Lambert [Lam07]. We then notice that both the genealogies of the time changed Feller diffusion and of the time changed Feller diffusion conditioned on non extinction are given by the same Kingman's coalescent. On the contrary, the genealogy of the time changed α-stable CB-process is a $Beta(2-\alpha, \alpha)$-coalescent, whereas the genealogy of the time changed α-stable CB-process conditioned on non-extinction is a $Beta(2-\alpha, \alpha-1)$-coalescent. We stress that for any $\alpha \in (1, 2)$ and any borelian B of $[0, 1]$, we have $Beta(2-\alpha, \alpha-1)(B) \geq Beta(2-\alpha, \alpha)(B)$. This may be interpreted as the additional reproduction events needed for the process to be never extinct.

2.4.3 Proofs.

Proof of Lemma 2.5. Let $(\Pi'(t), t \geq 0)$ be a Λ-coalescent on $\mathcal{P}^0_\infty$. Let $n \geq 1$, we may express the jump rate of $(\Pi'_{|[n]}(t), t \geq 0)$ from $0_{[n]}$ to π by

$$q'_\pi = \begin{cases} 0 \text{ if } \pi \text{ has more than one non-trivial block} \\ \\ \int_{[0,1]} x^k(1-x)^{n+1-k} x^{-2} \Lambda(dx) \text{ if the non trivial block has } k \text{ elements.} \end{cases}$$

Consider now a M-coalescent, denoting by q_π the jump rate from $0_{[n]}$ to π, we have

$$q_\pi = \begin{cases} 0 \text{ if } \pi \text{ has more than one non-trivial block} \\ \\ \int_{[0,1]} x^k(1-x)^{n-k} x^{-2} \Lambda_1(dx) \text{ if } \pi_0 = \{0\} \text{ and the non trivial block has } k \text{ elements} \\ \\ \int_{[0,1]} x^{k-1}(1-x)^{n+1-k} x^{-1} \Lambda_0(dx) \text{ if } \#\pi_0 = k. \end{cases}$$

Since the law of a Λ-coalescent is entirely described by the family of the jump rates of its restriction on $[n]$ from $0_{[n]}$ to π for π belonging to $\mathcal{P}^0_n$ (see Section 4.2 of [Ber06]), the processes Π and Π' have the same law if and only if for all $n \geq 0$ and $\pi \in \mathcal{P}^0_n$, we have $q_\pi = q'_\pi$, that is if and only if $(1-x)\Lambda_0(dx) = \Lambda_1(dx)$. □

Proof of Corollary 2.7. Since $C^{-1}(C(t)) = t$,

$$\mathbb{E}\left[\int_{[0,1]^p} f(x_1, ..., x_p) R_t(dx_1)...R_t(dx_p)\right] = \mathbb{E}\left[\int_{[0,1]^p} f(x_1, ..., x_p) R_{C^{-1}(C(t))}(dx_1)...R_{C^{-1}(C(t))}(dx_p)\right].$$

Then, using the independence between $R_{C^{-1}}$ and C, the right hand side above is also equal to :

$$\int \mathbb{P}(C(t) \in ds)\ \mathbb{E}\left[\int_{[0,1]^p} f(x_1, ..., x_p) R_{C^{-1}(s)}(dx_1)...R_{C^{-1}(s)}(dx_p)\right].$$

Using Corollary 2.6 and choosing $(\Pi(t), t \geq 0)$ independent of $(C(t), t \geq 0)$, we find :

$$\begin{aligned} &\int \mathbb{P}(C(t) \in ds)\ \mathbb{E}\left[\int_{[0,1]^p} f(x_1, ..., x_p) R_{C^{-1}(s)}(dx_1)...R_{C^{-1}(s)}(dx_p)\right] \\ &= \int \mathbb{P}(C(t) \in ds)\ \mathbb{E}\left[\int_{[0,1]^{p+1}} f(x_{\alpha_{\Pi(s)}(1)}, ..., x_{\alpha_{\Pi(s)}(p)}) \delta_0(dx_0) dx_1...dx_p\right] \\ &= \mathbb{E}\left[\int_{[0,1]^{p+1}} f(x_{\alpha_{\Pi(C(t))}(1)}, ..., x_{\alpha_{\Pi(C(t))}(p)}) \delta_0(dx_0) dx_1...dx_p\right]. \end{aligned}$$

□

Remark 4 *Notice the crucial rôle of the independence in order to establish Corollary 2.7. When this property fails, as in the case (ii), the question of describing the fixed time genealogy of the α-stable CB or CBI remains open. We refer to the discussion in Section 2.2 of Berestycki et. al [BB09].*

2.5 Proof of Theorem 2.3 and Proposition 2.4

We first deal with Theorem 2.3. The proof of Proposition 2.4 is rather technical ans is postponed at the end of this Section. In order to get the connection between the two measure-valued processes $(R_t, t \geq 0)$ and $(M_t, t \geq 0)$, we may follow the ideas of Birkner *et al.* [BBC+05] and rewrite the generator of the process $(M_t, t \geq 0)$ using the "polar coordinates" : for any $\eta \in \mathcal{M}_f$, we define

$$z := |\eta| \text{ and } \rho := \frac{\eta}{|\eta|}.$$

The proof relies on five lemmas. Lemma 2.9 establishes that the law of a generalized Fleming-Viot process with immigration is entirely determined by the generator $\mathcal{F}$ on the test functions of the form $\rho \mapsto \langle \phi, \rho \rangle^m$ with ϕ a measurable non-negative bounded map and $m \in \mathbb{N}$. Lemmas 2.10, 2.11 and 2.13 allow us to study the generator $\mathcal{L}$ on the class of functions of the type $F : \eta \mapsto \frac{1}{|\eta|^m}\langle \phi, \eta \rangle^m$. Lemma 2.12 (lifted from Lemma 3.5 of [BBC+05]) relates stable Lévy-measures and Beta-measures. We end the proof using results on time change by the inverse of an additive functional. We conclude thanks to a result due to Volkonskiĭ in [Vol58] about the generator of a time-changed process.

Lemma 2.9 *The following martingale problem is well-posed : for any function f of the form :*

$$(x_1, ..., x_p) \mapsto \prod_{i=1}^{p} \phi(x_i)$$

with ϕ a non-negative measurable bounded map and $p \geq 1$, the process

$$G_f(\rho_t) - \int_0^t \mathcal{F}G_f(\rho_s)ds$$

is a martingale.

Proof. Only the uniqueness has to be checked. We shall establish that the martingale problem of the statement is equivalent to the following martingale problem : for any continuous function f on $[0,1]^p$, the process

$$G_f(\rho_t) - \int_0^t \mathcal{F}G_f(\rho_s)ds$$

is a martingale. This martingale problem is well posed, see Proposition 5.2 of [Fou11]. Notice that we can focus on continuous and symmetric functions since for any continuous f, $G_f = G_{\tilde{f}}$ with $\tilde{f}$ the symmetrized version of f. Moreover, by the Stone-Weierstrass theorem, any symmetric continuous function f from $[0,1]^p$ to $\mathbb{R}$ can be uniformly approximated by linear combination of functions of the form $(x_1, ..., x_p) \mapsto \prod_{i=1}^{p} \phi(x_i)$ for some function ϕ continuous on $[0,1]$. We now take f symmetric and continuous, and let f_k be an approximating sequence. Plainly, we have

$$|G_{f_k}(\rho) - G_f(\rho)| \leq ||f_k - f||_\infty$$

Assume that $(\rho_t, t \geq 0)$ is a solution of the martingale problem stated in the lemma. Since the map $h \mapsto G_h$ is linear, the process

$$G_{f_k}(\rho_t) - \int_0^t \mathcal{F}G_{f_k}(\rho_s)ds$$

is a martingale for each $k \geq 1$. We want to prove that the process

$$G_f(\rho_t) - \int_0^t \mathcal{F}G_f(\rho_s)ds$$

is a martingale, knowing it holds for each f_k. We will show the following convergence

$$\mathcal{F}G_{f_k}(\rho) \underset{k\to\infty}{\longrightarrow} \mathcal{F}G_f(\rho) \text{ uniformly in } \rho.$$

Recall expressions (1') and (2') in Subsection 2.2.2, one can check that the following limits are uniform in the variable ρ

$$\sum_{1\leq i<j\leq p} \int_{[0,1]^p} [f_k(\mathrm{x}^{i,j}) - f_k(\mathrm{x})]\rho^{\otimes p}(dx) \underset{k\to\infty}{\longrightarrow} \sum_{1\leq i<j\leq p} \int_{[0,1]^p} [f(\mathrm{x}^{i,j}) - f(\mathrm{x})]\rho^{\otimes p}(dx)$$

and

$$\sum_{1\leq i\leq m} \int_{[0,1]^p} [f_k(\mathrm{x}^{0,i}) - f_k(\mathrm{x})]\rho^{\otimes p}(dx) \underset{k\to\infty}{\longrightarrow} \sum_{1\leq i\leq p} \int_{[0,1]^p} [f(\mathrm{x}^{0,i}) - f(\mathrm{x})]\rho^{\otimes p}(dx).$$

We have now to deal with the terms (3') and (4'). In order to get that the quantity

$$\int_0^1 \nu(dr) \int_0^1 [G_{f_k}((1-r)\rho + r\delta_a) - G_{f_k}(\rho)]\rho(da)$$

converges toward

$$\int_0^1 \nu(dr) \int_0^1 [G_f((1-r)\rho + r\delta_a) - G_f(\rho)]\rho(da),$$

we compute

$$\langle f_k - f, ((1-r)\rho + r\delta_a)^{\otimes p}\rangle - \langle f_k - f, \rho^{\otimes p}\rangle.$$

Since the function $f_k - f$ is symmetric, we may expand the p-fold product $\langle f_k - f, ((1-r)\rho + r\delta_a)^{\otimes p}\rangle$, this yields

$$\begin{aligned}
&\langle f_k - f, ((1-r)\rho + r\delta_a)^{\otimes p}\rangle - \langle f_k - f, \rho^{\otimes p}\rangle \\
&\qquad = \sum_{i=0}^{p} \binom{p}{i} r^i (1-r)^{p-i} \left(\langle f_k - f, \rho^{\otimes p-i} \otimes \delta_a^{\otimes i}\rangle - \langle f_k - f, \rho^{\otimes p}\rangle\right) \\
&\qquad = pr(1-r)^{p-1} \left(\langle f_k - f, \rho^{\otimes p-1} \otimes \delta_a\rangle - \langle f_k - f, \rho^{\otimes p}\rangle\right) \\
&\qquad\quad + \sum_{i=2}^{p} \binom{p}{i} r^i (1-r)^{p-i} \left(\langle f_k - f, \rho^{\otimes p-i} \otimes \delta_a^{\otimes i}\rangle - \langle f_k - f, \rho^{\otimes p}\rangle\right).
\end{aligned}$$

We use here the notation

$$\langle g, \mu^{\otimes m-i} \otimes \delta_a^{\otimes i}\rangle := \int g(x_1, ..., x_{m-i}, \underbrace{a, ..., a}_{i \text{ terms}})\mu(dx_1)...\mu(dx_{m-i}).$$

Therefore, integrating with respect to ρ, the first term in the last equality vanishes and we get

$$\left|\int_0^1 \rho(da) \left(G_{f-f_k}((1-r)\rho + r\delta_a) - G_{f-f_k}(\rho)\right)\right| \leq 2^{p+1}||f - f_k||_\infty r^2$$

where $||f_k - f||_\infty$ denotes the supremum of the function $|f_k - f|$. Recall that the measure ν_1 verifies $\int_0^1 r^2\nu_1(dr) < \infty$, moreover the quantity $||f_k - f||_\infty$ is bounded. Thus appealing to the Lebesgue Theorem, we get the sought-after convergence. Same arguments hold for the immigration part (4') of the operator $\mathcal{F}$. Namely we have

$$|G_{f-f_k}((1-r)\rho + r\delta_0) - G_{f-f_k}(\rho)| \leq 2^{p+1} r ||f_k - f||_\infty$$

and the measure ν_0 satisfies $\int_0^1 r\nu_0(dr) < \infty$. Combining our results, we obtain

$$|\mathcal{F}G_{f_k}(\rho) - \mathcal{F}G_f(\rho)| \leq C||f - f_k||_\infty$$

for a positive constant C independent of ρ. Therefore the sequence of martingales $G_{f_k}(\rho_t) - \int_0^t \mathcal{F}G_{f_k}(\rho_s)ds$ converges toward

$$G_f(\rho_t) - \int_0^t \mathcal{F}G_f(\rho_s)ds,$$

which is then a martingale. □

Lemma 2.10 *Assume that $\hat{\nu_0} = \hat{\nu_1} = 0$ the generator $\mathcal{L}$ of $(M_t, t \geq 0)$ is reduced to the expressions (1) and (2) :*

$$\mathcal{L}F(\eta) = \sigma^2/2 \int_0^1 \int_0^1 \eta(da)\delta_a(db) F''(\eta; a, b) + \beta F'(\eta; 0)$$

Let ϕ be a measurable bounded function on $[0,1]$ and F be the map $\eta \mapsto G_f(\rho) := \langle f, \rho^{\otimes m}\rangle$ with $f(x_1, ..., x_p) = \prod_{i=1}^p \phi(x_i)$. We have the following identity

$$|\eta|\mathcal{L}F(\eta) = \mathcal{F}G_f(\rho),$$

for $\eta \neq 0$, where $\mathcal{F}$ is the generator of a Fleming-Viot process with immigration with reproduction rate $c_1 = \sigma^2$ and immigration rate $c_0 = \beta$, see expressions (1') and (2').

Proof. By the calculations in Section 4.3 of Etheridge [Eth00] (but in a non-spatial setting, see also the proof of Theorem 2.1 p. 249 of Shiga [Shi90]), we get :

$$\begin{aligned}\frac{\sigma^2}{2} \int_0^1 \int_0^1 \eta(da)\delta_a(db) F''(\eta; a, b) &= |\eta|^{-1} \frac{\sigma^2}{2} \int_0^1 \int_0^1 \frac{\partial^2 G_f}{\partial\rho(a)\partial\rho(b)}(\rho)[\delta_a(db) - \rho(db)]\rho(da) \\ &= |\eta|^{-1}\sigma^2 \sum_{1\leq i<j\leq m} \int_{[0,1]^p} [f(\mathrm{x}^{i,j}) - f(\mathrm{x})]\rho^{\otimes m}(dx).\end{aligned}$$

We focus now on the immigration part. We take f a function of the form $f : (x_1, ..., x_m) \mapsto \prod_{i=1}^m \phi(x_i)$ for some function ϕ, and consider $F(\eta) := G_f(\rho) = \langle f, \rho^{\otimes m}\rangle$. We may compute :

$$\begin{aligned}F(\eta + h\delta_a) - F(\eta) &= \left\langle \phi, \frac{\eta + h\delta_a}{z+h} \right\rangle^m - \langle \phi, \rho\rangle^m \\ &= \sum_{j=2}^m \binom{m}{j} \left(\frac{z}{z+h}\right)^{m-j} \left(\frac{h}{z+h}\right)^j [\langle\phi,\rho\rangle^{m-j}\phi(a)^j - \langle\phi,\rho\rangle^m] \quad (2.11) \\ &\quad + m\left(\frac{z}{z+h}\right)^{m-1}\left(\frac{h}{z+h}\right)[\langle\phi,\rho\rangle^{m-1}\phi(a) - \langle\phi,\rho\rangle^m]\,. \quad (2.12)\end{aligned}$$

We get that :

$$F'(\eta; a) = \frac{m}{z}\left[\phi(a)\langle\phi,\rho\rangle^{m-1} - \langle\phi,\rho\rangle^m\right].$$

Thus,

$$F'(\eta;0) = |\eta|^{-1} \sum_{1\le i\le m} \int_{[0,1]^p} [f(\mathrm{x}^{0,i}) - f(\mathrm{x})]\rho^{\otimes m}(d\mathrm{x})$$

and

$$\int F'(\eta;a)\eta(da) = 0 \tag{2.13}$$

for such function f. This proves the Lemma.□

This first lemma will allow us to prove the case (i) of Theorem 2.3. We now focus on the case (ii). Assuming that $\sigma^2 = \beta = 0$, the generator of $(M_t, t \ge 0)$ reduces to

$$\mathcal{L}F(\eta) = \mathcal{L}_0 F(\eta) + \mathcal{L}_1 F(\eta) \tag{2.14}$$

where, as in equations (3) and (4) of Subsection 2.2.1,

$$\begin{aligned}
\mathcal{L}_0 F(\eta) &= \int_0^\infty \hat{\nu_0}(dh)[F(\eta + h\delta_0) - F(\eta)] \\
\mathcal{L}_1 F(\eta) &= \int_0^1 \eta(da) \int_0^\infty \hat{\nu_1}(dh)[F(\eta + h\delta_a) - F(\eta) - hF'(\eta,a)].
\end{aligned}$$

The following lemma is a first step to understand the infinitesimal evolution of the non-markovian process $(R_t, t \ge 0)$ in the purely discontinuous case.

Lemma 2.11 *Let f be a continuous function on $[0,1]^p$ of the form $f(x_1,...,x_p) = \prod_{i=1}^p \phi(x_i)$ and F be the map $\eta \mapsto G_f(\rho) = \langle \phi, \rho\rangle^p$. Recall the notation $\rho := \eta/|\eta|$ and $z = |\eta|$. We have the identities :*

$$\begin{aligned}
\mathcal{L}_0 F(\eta) &= \int_0^\infty \hat{\nu_0}(dh) \left[G_f\left([1 - \frac{h}{z+h}]\rho + \frac{h}{z+h}\delta_0 \right) - G_f(\rho) \right] \\
\mathcal{L}_1 F(\eta) &= z \int_0^\infty \hat{\nu_1}(dh) \int_0^1 \rho(da) \left[G_f\left([1 - \frac{h}{z+h}]\rho + \frac{h}{z+h}\delta_a \right) - G_f(\rho) \right].
\end{aligned}$$

Proof. The identity for $\mathcal{L}_0$ is plain, we thus focus on $\mathcal{L}_1$. Combining Equation (2.13) and the term (2.12) we get

$$\int_0^1 \rho(da) \left[m \left(\frac{z}{z+h} \right)^{m-1} \left(\frac{h}{z+h} \right) [\langle \phi, \rho\rangle^{m-1}\phi(a) - \langle \phi, \rho\rangle^m] - hF'(\eta;a) \right] = 0.$$

We easily check from the terms of (2.11) that the map $h \mapsto \int_0^1 \rho(da)[F(\eta + h\delta_a) - F(\eta) - hF'(\eta,a)]$ is integrable with respect to the measure $\hat{\nu}_1$. This allows us to interchange the integrals and yields :

$$\mathcal{L}_1 F(\eta) = z \int_0^\infty \hat{\nu_1}(dh) \int_0^1 \rho(da) \left[G_f\left(\frac{\eta + h\delta_a}{z+h} \right) - G_f(\rho) \right]. \tag{2.15}$$

□

The previous lemma leads us to study the images of the measures $\hat{\nu}_0$ and $\hat{\nu}_1$ by the map $\phi_z : h \mapsto r := \frac{h}{h+z}$, for every $z > 0$. Denote $\lambda_z^0(dr) = \hat{\nu}_0 \circ \phi_z^{-1}$ and $\lambda_z^1(dr) = \hat{\nu}_1 \circ \phi_z^{-1}$. The following lemma is lifted from Lemma 3.5 of [BBC+05].

Lemma 2.12 *There exist two measures ν_0, ν_1 such that $\lambda_z^0(dr) = s_0(z)\nu_0(dr)$ and $\lambda_z^1(dr) = s_1(z)\nu_1(dr)$ for some maps s_0, s_1 from $\mathbb{R}_+$ to $\mathbb{R}$ if and only if for some $\alpha \in (0,2), \alpha' \in (0,1)$ and $c, c' > 0$:*

$$\hat{\nu_1}(dx) = cx^{-1-\alpha}dx,\ \hat{\nu_0}(dx) = c'x^{-1-\alpha'}dx.$$

In this case :

$$s_1(z) = z^{-\alpha},\ \nu_1(dr) = r^{-2}cBeta(2-\alpha,\alpha)(dr)$$

and

$$s_0(z) = z^{-\alpha'},\ \nu_0(dr) = r^{-1}c'Beta(1-\alpha',\alpha')(dr).$$

Proof. The necessary part is given by the same arguments as in Lemma 3.5 of [BBC+05]. We focus on the sufficient part. Assuming that $\hat{\nu_0}, \hat{\nu_1}$ are as above, we have

- $\lambda_z^1(dr) = cz^{-\alpha}r^{-1-\alpha}(1-r)^{-1+\alpha}dr = z^{-\alpha}r^{-2}cBeta(2-\alpha,\alpha)(dr)$, and thus $s_1(z) = z^{-\alpha}$.
- $\lambda_z^0(dr) = c'z^{-\alpha'}r^{-1-\alpha'}(1-r)^{-1+\alpha'}dr = z^{-\alpha'}r^{-1}c'Beta(1-\alpha',\alpha')(dr)$ and thus $s_0(z) = z^{-\alpha'}$. $\square$

The next lemma allows us to deal with the second statement of Theorem 2.3.

Lemma 2.13 *Assume that $\sigma^2 = \beta = 0$, $\hat{\nu_0}(dh) = ch^{-\alpha}1_{h>0}dh$ and $\hat{\nu_1}(dh) = ch^{-1-\alpha}1_{h>0}dh$. Let f be a function on $[0,1]^p$ of the form $f(x_1,...,x_p) = \prod_{i=1}^p \phi(x_i)$, and F be the map $\eta \mapsto G_f(\rho)$. We have*

$$|\eta|^{\alpha-1}\mathcal{L}F(\eta) = \mathcal{F}G_f(\rho),$$

for $\eta \neq 0$, where $\mathcal{F}$ is the generator of a M-Fleming-Viot process with immigration, with $M = (c'Beta(2-\alpha,\alpha-1), cBeta(2-\alpha,\alpha))$, see expressions $(3')$, $(4')$.

Proof. Recall Equation (2.14) :

$$\mathcal{L}F(\eta) = \mathcal{L}_0F(\eta) + \mathcal{L}_1F(\eta)$$

Recall from Equation (13) that we have $\int_0^1 F'(\eta;a)\eta(da) = 0$ for $F(\eta) = G_f(\rho)$. Applying Lemma 2.11 and Lemma 2.12, we get that in the case $\sigma^2 = \beta = 0$ and $\hat{\nu_1}(dx) = cx^{-1-\alpha}dx, \hat{\nu_0}(dx) = c'x^{-1-\alpha'}dx$:

$$\begin{aligned}\mathcal{L}F(\eta) = \mathcal{L}G_f(\rho) = \quad & s_0(z)\int_0^1 r^{-1}c'Beta(1-\alpha',\alpha')(dr)[G_f((1-r)\rho + r\delta_0) - G_f(\rho)] \\ & + zs_1(z)\int_0^1 r^{-2}cBeta(2-\alpha,\alpha)(dr)\int_0^1 \rho(da)[G_f((1-r)\rho + r\delta_a) - G_f(\rho)].\end{aligned}$$

Recalling the expressions (3'), (4'), the factorization $h(z)\mathcal{L}F(\eta) = \mathcal{F}G(\rho)$ holds for some function h if

$$s_0(z) = zs_1(z),$$

if $\alpha' = \alpha - 1$. In that case, $h(z) = z^{\alpha-1}$. $\square$

We are now ready to prove Theorem 2.3. To treat the case (i), replace α by 2 in the sequel. The process $(Y_t, R_t)_{t\geq 0}$ with lifetime τ has the Markov property. The additive functional $C(t) = \int_0^t \frac{1}{Y_s^{\alpha-1}}ds$ maps $[0,\tau)$ to $[0,\infty)$. From Theorem 65.9 of [Sha88] and Proposition 2.2, the process $(Y_{C^{-1}(t)}, R_{C^{-1}(t)})_{t\geq 0}$ is a strong Markov process with infinite lifetime. Denote by $\mathcal{U}$ the generator of $(Y_t, R_t)_{t\geq 0}$. As explained in Birkner et al. [BBC+05] (Equation (2.6) p314), the law of $(Y_t, R_t)_{t\geq 0}$ is characterized by $\mathcal{U}$ acting on the following class of test functions :

$$(z,\rho)\in\mathbb{R}_+\times\mathcal{M}_1\mapsto F(z,\rho):=\psi(z)\langle\phi,\rho\rangle^m$$

for ϕ a non-negative measurable bounded function on $[0,1]$, $m\geq 1$ and ψ a twice differentiable non-negative map. Theorem 3 of Volkonskiĭ, see [Vol58] (or Theorem 1.4 Chapter 6 of [EK86]) states that the Markov process with generator

$$\tilde{\mathcal{U}}F(z,\rho):=z^{\alpha-1}\mathcal{U}F(z,\rho)$$

coincides with $(Y_{C^{-1}(t)},R_{C^{-1}(t)})_{t\geq 0}$. We establish now that $(R_{C^{-1}(t)},t\geq 0)$ is a Markov process with the same generator as the Fleming-Viot processes involved in Theorem 2.3. Let $G(z,\rho)=G_f(\rho)=\langle\phi,\rho\rangle^m$ (taking $f:(x_1,...,x_m)\mapsto\prod_{i=1}^m\phi(x_i)$). In both cases (i) and (ii) of Theorem 2.3, we have :

$$\begin{aligned}z^{\alpha-1}\mathcal{U}G(z,\rho)&=z^{\alpha-1}\mathcal{L}F(\eta)\text{ with }F:\eta\mapsto G_f(\rho)\\&=\mathcal{F}G_f(\rho).\end{aligned}$$

First equality holds since we took $\psi\equiv 1$ and the second uses Lemma 2.10 and Lemma 2.13. Since it does not depend on z, the process $(R_{C^{-1}(t)},t\geq 0)$ is a Markov process, moreover it is a generalized Fleming-Viot process with immigration with parameters as stated. □

Proof of Proposition 2.4. Let $(Y_t)_{t\geq 0}$ be a Feller branching diffusion with continuous immigration with parameters (σ^2,β). Consider an independent M-Fleming-Viot $(\rho_t,t\geq 0)$ with $M=(\beta\delta_0,\sigma^2\delta_0)$. We first establish that $(Y_t\rho_{C(t)},0\leq t<\tau)$ has the same law as the measure-valued branching process $(M_t,0\leq t<\tau)$. Recall that $\mathcal{L}$ denote the generator of $(M_t,t\geq 0)$ (here only the terms (1) and (2) are considered). Consider $F(\eta):=\psi(z)\langle\phi,\rho\rangle^m$ with $z=|\eta|$, ψ a twice differentiable map valued in $\mathbb{R}_+$ and ϕ a non-negative bounded measurable function. Note that the generator acting on such functions F characterizes the law of $(M_{t\wedge\tau},t\geq 0)$. First we easily obtain that

$$\begin{aligned}F'(\eta;0)&=\psi'(z)\langle\phi,\rho\rangle^m+m\frac{\psi(z)}{z}[\phi(0)\langle\phi,\rho\rangle^{m-1}-\langle\phi,\rho\rangle^m],\\F''(\eta;a,b)&=\psi''(z)\langle\phi,\rho\rangle^m+m\frac{\psi'(z)}{z}\left[(\phi(b)+\phi(a))\langle\phi,\rho\rangle^{m-1}-2\langle\phi,\rho\rangle^m\right]\\&\quad+m\frac{\psi(z)}{z^2}\left[(m-1)\phi(a)\phi(b)\langle\phi,\rho\rangle^{m-2}-m(\phi(a)+\phi(b))\langle\phi,\rho\rangle^{m-1}+(m+1)\langle\phi,\rho\rangle^m\right].\end{aligned}$$

Simple calculations yield,

$$\begin{aligned}\mathcal{L}F(\eta)&=\left[z\left(\frac{\sigma^2}{2}\psi''(z)\right)+\beta\psi'(z)\right]\langle\phi,\rho\rangle^m\\&\quad+\frac{\psi(z)}{z}\left[\sigma^2\frac{m(m-1)}{2}\left(\langle\phi^2,\rho\rangle\langle\phi,\rho\rangle^{m-2}-\langle\phi,\rho\rangle^m\right)+\beta m\left(\phi(0)\langle\phi,\rho\rangle^{m-1}-\langle\phi,\rho\rangle^m\right)\right].\end{aligned}$$

We recognize in the first line the generator of $(Y_t,t\geq 0)$ and in the second, $\frac{1}{z}\mathcal{F}G_f(\rho)$ with $f(x_1,...,x_m)=\prod_{i=1}^m\phi(x_i)$ and $c_0=\beta,c_1=\sigma^2$. We easily get that this is the generator of the Markov process $(Y_t\rho_{C(t)},t\geq 0)$ with lifetime τ. We conclude that it has the same law as $(M_{t\wedge\tau},t\geq 0)$. We rewrite this equality in law as follows :

$$(Y_t\ \rho_{C(t)},0\leq t<\tau)\overset{law}{=}(|M_t|\ R_{C^{-1}(C(t))},0\leq t<\tau),\tag{2.16}$$

with C defined by $C(t)=\int_0^t|M_s|^{-1}ds$ for $0\leq t<\tau$ on the right hand side. Since $(C(t),t\geq 0)$ and $(\rho_t,t\geq 0)$ are independent on the left hand side and the decomposition in (2.16) is unique, we have also $(C(t), 0 \leq t < \tau)$ and $(R_{C^{-1}(t)}, 0 \leq t < \tau)$ independent on the right hand side.

Concerning the case (ii) of Theorem 2.3, we easily observe that the presence of jumps implies that such a decomposition of the generator cannot hold. See for instance Equation (2.7) of [BBC^{+}05] p344. The processes $(R_{C^{-1}(t)}, t \geq 0)$ and $(Y_t, \geq 0)$ are not independent. $\square$

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Chapitre 3

Generalized Fleming-Viot processes with immigration and stochastic flows of partitions

3.1 Introduction

Originally, Fleming-Viot processes were defined in 1979 in [FV79] to model the genetic phenomenon of allele drift. They now form an important sub-class of measure-valued processes, which have received significant attention in the literature. Donnelly and Kurtz, in 1996, established in [DK96] a duality between the classical Fleming-Viot process and the Kingman coalescent (defined in [Kin82]). Shortly after, the class of coalescent processes was considerably generalized by assuming that multiple coagulations may happen simultaneously. We refer to Chapter 3 of [Ber09] for an introduction to the Λ-coalescents and to the seminal papers [MS01b], [Sch00a] for the definition of the general exchangeable coalescent processes (also called Ξ-coalescents). An infinite particle representation set up in 1999, allowed Donnelly and Kurtz, in [DK99] to define a generalized Fleming-Viot process by duality with the Λ-coalescent. In 2003, Bertoin and Le Gall started from another point of view and introduced, in [BLG03], a stochastic flow of bridges which encodes simultaneously an exchangeable coalescent process and a continuous population model, the so-called generalized Fleming-Viot process. Finally, in 2009, Birkner *et al.* in [BBM$^+$09] have adapted the same arguments as Donnelly-Kurtz for the case of the Ξ-coalescent.

In Chapter 1, a larger class of coalescents called distinguished coalescents were defined, in order to incorporate immigration in the underlying population. The purpose of this article is to define by duality the class of generalized Fleming-Viot processes with immigration. Imagine an infinite haploid population with immigration, identified by the set $\mathbb{N}$. This means that each individual has at most one parent in the population at the previous generation ; indeed, immigration implies that some individuals may have parents outside this population (they are children of immigrants). To encode the arrival of new immigrants, we consider the external integer 0 as the generic parent of immigrants. We shall then work with partitions of $\mathbb{Z}_+$, the so-called *distinguished* partitions. Our approach will draw both on the works of Bertoin-Le Gall and of Donnelly-Kurtz. Namely, in the same vein as Bertoin and Le Gall's article [BLG03], we define a *stochastic flow of partitions* of $\mathbb{Z}_+$, denoted by $(\Pi(s,t), -\infty < s \le t < \infty)$. The dual flow $(\hat{\Pi}(t), t \ge 0) := (\Pi(-t, 0), t \ge 0)$ shall encode an infinite haploid population model with immigration forward in time. Namely, for any individual $i \geq 1$ alive at the initial time 0, the set $\hat{\Pi}_i(t)$ shall represent the descendants of i, and $\hat{\Pi}_0(t)$ the descendants of the generic immigrant, at time t. We stress that the evolution mechanism involves both multiple reproduction and immigration. The genealogy of this population model is precisely a distinguished exchangeable coalescent. Our method is close to the "modified" lookdown construction of Donnelly-Kurtz, but differs from (and simplifies) the generalization given by Birkner et al. in [BBM$^+$09]. In the same vein as Fleming and Viot's fundamental article [FV79], we consider the type carried by each individual at the initial time. Denote by U_i the type of the individual $i \geq 1$, and distinguish the type of the generic immigrant by fixing $U_0 = 0$. The study of the evolution of frequencies as time passes, leads us to define and study the so-called generalized Fleming-Viot process with immigration (GFVI for short) denoted in the sequel by $(\rho_t, t \geq 0)$. This process will be explicitly related to the *forward partition-valued process* $(\hat{\Pi}(t), t \geq 0)$. When the process $(\hat{\Pi}(t), t \geq 0)$ is absorbed in the trivial partition $(\{\mathbb{Z}_+\}, \emptyset, ...)$, all individuals are immigrant children after a certain time. We shall discuss conditions entailing the occurrence of this event.

The rest of the paper is organized as follows. In Section 2 (Preliminaries), we recall some basic facts on distinguished partitions. We introduce a coagulation operator *coag* and describe how a population may be encoded by distinguished partitions forward in time. Some properties related to exchangeable sequences and partitions are then presented. We recall the definition of an exchangeable distinguished coalescent and define a stochastic flow of partitions using a Poisson random measure on the space of partitions. In Section 3, we study the dual flow and the embedded population. Adding initially a type to each individual, the properties of exchangeability of Section 2 allow us to define the generalized Fleming-Viot process with immigration. We show that any generalized Fleming-Viot process with immigration is a Feller process. Arguing then by duality we determine the generator of any GFVI on a space of functionals which forms a core. In Section 4, we give a sufficient condition for the extinction of the initial types. Thanks to the duality established in Section 3, the extinction corresponds to the coming down from infinity of the distinguished coalescent.

3.2 Preliminaries

We start by recalling some basic definitions and results about exchangeable distinguished partitions which are developed in Chapter 1. For every $n \geq 0$, we denote by $[\,n\,]$ the set $\{0, ..., n\}$ and call $\mathcal{P}_n^0$ the set of partitions of $[\,n\,]$. The space $\mathcal{P}_\infty^0$ is the set of partitions of $[\,\infty\,] := \mathbb{Z}_+$. By convention the blocks are listed in the increasing order of their least element and we denote by π_i the i-th block of the partition π. The first block of π, which contains 0, is thus π_0 and is viewed as distinguished. It shall represent descendants of an immigrant. An element of $\mathcal{P}_\infty^0$ is then called a distinguished partition. We endow the space $\mathcal{P}_\infty^0$ with a distance, that makes it compact, defined by

$$d(\pi, \pi') = (1 + \max\{n \geq 0; \pi_{|\,[\,n\,]} = \pi'_{|\,[\,n\,]}\})^{-1}.$$

The notation $i \overset{\pi}{\sim} j$ means that i and j are in the same block of π. An exchangeable distinguished partition is a random element π of $\mathcal{P}_\infty^0$ such that for all permutations σ satisfying $\sigma(0) = 0$, the partition $\sigma\pi$ defined by

$$i \overset{\sigma\pi}{\sim} j \text{ if and only if } \sigma(i) \overset{\pi}{\sim} \sigma(j),$$

has the same law as π. If π is a distinguished exchangeable partition, the asymptotic frequency

$$|\pi_i| := \lim_{n\to\infty} \frac{\#(\pi_i \cap [\,n\,])}{n}$$

exists for every $i \geq 0$, almost-surely. We denote by $|\pi|^{\downarrow}$ the sequence of asymptotic frequencies $(|\pi_i|, i \geq 0)$ after a decreasing rearrangement apart from $|\pi_0|$. Thanks to Kingman's correspondence, the law of $|\pi|^{\downarrow}$ determines completely that of π (see Theorem 1.5 in Chapter 1).

The operator *coag* is defined from $\mathcal{P}^0_\infty \times \mathcal{P}^0_\infty$ to $\mathcal{P}^0_\infty$ such that for $(\pi, \pi') \in \mathcal{P}^0_\infty \times \mathcal{P}^0_\infty$, the partition $coag(\pi, \pi')$ satisfies for all $i \geq 0$,

$$coag(\pi, \pi')_i = \bigcup_{j \in \pi'_i} \pi_j.$$

The operator *coag* is Lipschitz-continuous and associative in the sense that for any π, π' and π'' in $\mathcal{P}^0_\infty$

$$coag(\pi, coag(\pi', \pi'')) = coag(coag(\pi, \pi'), \pi'').$$

Moreover, the partition of $\mathbb{Z}_+$ into singletons, denoted by $0_{[\,\infty\,]}$, may be viewed as a neutral element for the operator *coag*, indeed

$$coag(\pi, 0_{[\,\infty\,]}) = coag(0_{[\,\infty\,]}, \pi) = \pi.$$

More generally, for every $n \geq 0$, we denote by $0_{[\,n\,]}$ the partition of $[\,n\,]$ into singletons. It should be highlighted that given two independent exchangeable distinguished partitions π, π', the partition $coag(\pi, \pi')$ is still exchangeable (see Lemma 4.3 in [Ber06]).

3.2.1 Partitions of the population and exchangeability

We explain how the formalism of partitions may be used to describe a population with immigration as time goes forward. As in the Introduction, imagine an infinite haploid population with immigration, identified by the set $\mathbb{N}$ evolving forward in time. An additional individual 0 is added and plays the role of a generic immigrant. The model may be described as follows : let $t_0 < t_1$,

– at time t_1 the families sharing the same ancestor at time t_0 form a distinguished partition $\pi^{(1)}$. The distinguished block $\pi^{(1)}_0$ comprises the children of immigrants,

– the indices of the blocks of $\pi^{(1)}$ are viewed as the ancestors living at time t_0. In other words, for any $j \geq 0$, the block $\pi^{(1)}_j$ is the offspring at time t_1 of the individual j living at time t_0.

Consider a time $t_2 > t_1$ and denote by $\pi^{(2)}$ the partition of the population at time t_2 such that the block $\pi^{(2)}_k$ comprises the descendants at time t_2 of the individual k, living at time t_1. Obviously, the set $\bigcup_{k\in\pi^{(1)}_j} \pi^{(2)}_k$ represents the descendants at time t_2 of the individual j at time t_0. Therefore the partition $coag(\pi^{(2)}, \pi^{(1)})$ encodes the descendants at time t_2 of the individuals living at time t_0.

For any fixed $\pi \in \mathcal{P}^0_\infty$, define the map α_π by

$$\alpha_\pi(k) := \text{the index of the block of } \pi \text{ containing } k.$$

Thus, in the population above, $\alpha_{\pi^{(1)}}(k)$ corresponds to the ancestor at time t_0 of the individual k in the population at time t_1. We have, by definition of the operator *coag*, the key equality

$$\alpha_{coag(\pi,\pi')} = \alpha_{\pi'} \circ \alpha_\pi.$$

Therefore, the ancestor living at time t_0 of the individual k at time t_2 is

$$\alpha_{\pi^{(1)}} \circ \alpha_{\pi^{(2)}}(k) = \alpha_{coag(\pi^{(2)},\pi^{(1)})}(k).$$

We give in the sequel some properties of the map α_π when π is an exchangeable distinguished partition. They will be useful to define a generalized Fleming-Viot process with immigration. We denote by $\mathcal{M}_1$ the space of probability measures on $[0,1]$. Let ρ be a random probability measure on $[0,1]$. We say that the exchangeable sequence $(U_i, i \geq 1)$ has de Finetti measure ρ, if conditionally given ρ the variables $(U_i, i \geq 1)$ are i.i.d. with law ρ. We make the key observation.

Lemma 3.1 *Let $(U_i, i \geq 1)$ be an infinite exchangeable sequence taking values in $[0,1]$, with de Finetti measure ρ and fix $U_0 = 0$. Let π be an independent distinguished exchangeable partition, then the infinite sequence $(U_{\alpha_\pi(k)}, k \geq 1)$ is exchangeable. Furthermore, its de Finetti measure is*

$$(1 - \sum_{i\geq 0} |\pi_i|)\rho + \sum_{i\geq 1} |\pi_i| \delta_{U_i} + |\pi_0|\delta_0.$$

Proof. The proof requires rather technical arguments and is given in the Appendix.

Remark 5 *Let $(U_i, i \geq 1)$ be an i.i.d. sequence with a continuous distribution ρ. Observing that*

$$k \overset{\pi}{\sim} l \Longleftrightarrow U_{\alpha_\pi(k)} = U_{\alpha_\pi(l)},$$

we can use the previous lemma to recover the (distinguished) paint-box structure of any (distinguished) exchangeable random partition, see Kingman [Kin78] or Theorem 2.1 in [Ber06] for the case with no distinguished block. Moreover, Lemma 3.1 yields another simple proof for the exchangeability of $coag(\pi, \pi')$ provided that π and π' are independent and both exchangeable.

3.2.2 Distinguished coalescents and flows of partitions

We start by recalling some basic facts on distinguished coalescents which are developed in Section 3 of Chapter 1. In particular, we recall the definition of a coagulation measure and its decomposition.

Distinguished coalescents and coagulation measure

Consider an infinite haploid population model with immigration, the population being identified with $\mathbb{N}$. Recall that a generic immigrant 0 is added to the population. We denote by $\Pi(s)$ the partition of the current population into families having the same ancestor s generations *earlier*. As explained before, individuals issued from the immigration form the distinguished block $\Pi_0(s)$ of $\Pi(s)$. When some individuals have the same ancestor at a generation s, they have the same ancestor at any previous generation. The following statement makes these ideas formal. We stress that time goes backward. An exchangeable distinguished coalescent is a Markov process $(\Pi(t), t \geq 0)$ valued in $\mathcal{P}^0_\infty$ such that given $\Pi(s)$

$$\Pi(s+t) \overset{Law}{=} coag(\Pi(s), \pi),$$

where π is an exchangeable distinguished partition independent of $\Pi(s)$, with a law depending only on t. A distinguished coalescent is called standard if $\Pi(0) = 0_{[\infty]}$. We emphasize that classical coagulations and coagulations with the distinguished block may happen simultaneously.
A distinguished coalescent is characterized by a measure μ on $\mathcal{P}^0_\infty$, called the coagulation measure, which fulfills the following conditions :

- μ is exchangeable, meaning here invariant under the action of the permutations σ of $\mathbb{Z}_+$ with finite support (i.e permuting only finitely many points), such that $\sigma(0) = 0$;
- $\mu(\{0_{[\infty]}\}) = 0$ and for all $n \geq 0$, $\mu(\pi \in \mathcal{P}^0_\infty : \pi_{|[n]} \neq 0_{[n]}) < \infty$.

To be more precise, let $(\Pi(t), t \geq 0)$ be a distinguished coalescent ; the coagulation measure μ is defined from the jump rates of the restricted processes $(\Pi_{|[n]}(t), t \geq 0)$ for $n \geq 0$. For every $\pi \in \mathcal{P}^0_n$, let q_π be the jump rate of $(\Pi_{|[n]}(t), t \geq 0)$ from $0_{[n]}$ to π and $\mathcal{P}^0_{\infty,\pi}$ be the set

$$\mathcal{P}^0_{\infty,\pi} := \{\pi' \in \mathcal{P}^0_\infty ; \pi'_{|[n]} = \pi\}.$$

We have by definition $\mu(\mathcal{P}^0_{\infty,\pi}) = q_\pi$. We will denote by $\mathcal{L}^*_n$ the generator of the continuous Markov chain $(\Pi_{|[n]}(t), t \geq 0)$. Let ϕ be a map from $\mathcal{P}^0_n$ to $\mathbb{R}$ and $\pi \in \mathcal{P}^0_n$, then

$$\mathcal{L}^*_n \phi(\pi) = \sum_{\pi' \in \mathcal{P}^0_n} q_{\pi'} [\phi(coag(\pi, \pi')) - \phi(\pi)].$$

Conversely for any coagulation measure μ, by the same arguments as in [Ber06] for the genuine coalescents, a distinguished coalescent with coagulation measure μ, is constructed using a Poisson random measure on the space $\mathbb{R}_+ \times \mathcal{P}^0_\infty$ with intensity $dt \otimes \mu$ (see Proposition 1.10 in Chapter 1). We mention that Theorem 1.12 in Chapter 1 yields a decomposition of a coagulation measure into a "Kingman part" and a "multiple collisions part". Let μ be a coagulation measure, then there exist c_0, c_1 non-negative real numbers and a measure ν on

$$\mathcal{P}^0_{\mathbf{m}} := \left\{ s = (s_0, s_1, ...);\ s_0 \geq 0, s_1 \geq s_2 \geq ... \geq 0, \sum_{i \geq 0} s_i \leq 1 \right\}$$

such that

$$\mu = c_0 \sum_{1 \leq i} \delta_{K(0,i)} + c_1 \sum_{1 \leq i < j} \delta_{K(i,j)} + \int_{\mathcal{P}^0_{\mathbf{m}}} \rho_{\mathbf{s}}(.) \nu(ds)$$

where $K(i,j)$ is the simple partition (meaning with at most one non-singleton block) with doubleton $\{i,j\}$ and ρ_s denotes the law of an **s**-distinguished paint-box (see Definition 3 in Chapter 1). The measure ν satisfies the condition

$$\int_{\mathcal{P}_{\mathbf{m}}} (s_0 + \sum_{i \geq 1} s_i^2) \nu(ds) < \infty.$$

Stochastic flow of partitions

We define a stochastic flow of partitions and give a construction from a Poisson random measure. The following definition may be compared with that of Bertoin and Le Gall's flows [BLG03].

Definition 3.2 *A flow of distinguished partitions is a collection of random variables* $(\Pi(s,t), -\infty < s \leq t < \infty)$ *valued in* $\mathcal{P}_\infty^0$ *such that :*

(i) For every $t \leq t'$*, the distinguished partition* $\Pi(t,t')$ *is exchangeable with a law depending only on* $t'-t$*.*
(ii) For every $t < t' < t''$, $\Pi(t,t'') = coag(\Pi(t,t'), \Pi(t',t''))$ *almost surely*
(iii) if $t'_1 < t'_2 < ... < t'_n$*, the distinguished partitions* $\Pi(t'_1,t'_2), ..., \Pi(t'_{n-1},t'_n)$ *are independent.*
(iv) $\Pi(0,0) = 0_{[\infty]}$ *and* $\Pi(t,t') \to 0_{[\infty]}$ *in probability when* $t'-t \to 0$*.*

The process $(\Pi(t), t \geq 0) := (\Pi(0,t), t \geq 0)$ is by definition a distinguished exchangeable coalescent. Given a coagulation measure μ, we introduce and study next a stochastic flow of partitions constructed from a Poisson random measure on $\mathbb{R} \times \mathcal{P}_\infty^0$ with intensity $dt \otimes \mu$. Instead of composing bridges as Bertoin and Le Gall in [BLG03], we coagulate directly the partitions replacing thus the operator of composition by the operator *coag*. For any partitions $\pi^{(1)}, ..., \pi^{(k)}$, we define recursively the partition $coag^k(\pi^{(1)}, ..., \pi^{(k)})$ by $coag^0 = 0_{[\infty]}$, $coag^1(\pi^{(1)}) = \pi^{(1)}$ and for all $k \geq 2$,

$$\begin{aligned} coag^k(\pi^{(1)}, ..., \pi^{(k)}) &:= coag\left(coag^{k-1}\left(\pi^{(1)}, ..., \pi^{(k-1)}\right), \pi^{(k)}\right) \\ &= coag^{k-1}\left(\pi^{(1)}, ..., \pi^{(k-2)}, coag\left(\pi^{(k-1)}, \pi^{(k)}\right)\right). \end{aligned}$$

Introduce a Poisson random measure $\mathcal{N}$ on $\mathbb{R} \times \mathcal{P}_\infty^0$ with intensity $dt \otimes \mu$ and for each $n \in \mathbb{N}$, let $\mathcal{N}_n$ be the image of $\mathcal{N}$ by the map $\pi \mapsto \pi_{|[n]}$. The condition $\mu(\pi_{|[n]} \neq 0_{[n]}) < \infty$ ensures that for all $s < t$ there are finitely many atoms of $\mathcal{N}_n$ in $]s,t] \times \mathcal{P}_n^0 \setminus 0_{[n]}$. We denote by $\{(t_1, \pi^{(1)}), (t_2, \pi^{(2)}), ..., (t_K, \pi^{(K)})\}$ these atoms with $K := \mathcal{N}_n(]s,t] \times \mathcal{P}_n^0 \setminus \{0_{[n]}\})$ and define

$$\Pi^n(s,t) := coag^K\left(\pi^{(1)}, ..., \pi^{(K)}\right).$$

It remains to establish the compatibility of the sequence of random partitions $(\Pi^n(s,t), n \in \mathbb{N})$, which means that for all $m \leq n$, $\Pi^n_{|[m]}(s,t) = \Pi^m(s,t)$. All non-trivial atoms of $\mathcal{N}_m$ are plainly non-trivial atoms of $\mathcal{N}_n$, and moreover the compatibility property of the operator *coag* with restrictions implies that

$$coag^K\left(\pi^{(1)}, ..., \pi^{(K)}\right)_{|[m]} = coag^K\left(\pi^{(1)}_{|[m]}, ..., \pi^{(K)}_{|[m]}\right).$$

Two cases may occur, either $\pi^{(i)}_{|[m]} = 0_{[m]}$ and does not affect the coagulation, or $\pi^{(i)}_{|[m]} \neq 0_{[m]}$ and is actually an atom of $\mathcal{N}_m$ on $]s,t] \times \mathcal{P}_m^0 \setminus \{0_{[m]}\}$. We then have the following identity :

$$\Pi^m(s,t) = \Pi^n_{|[m]}(s,t).$$

This compatibility property allows us to define a unique process $(\Pi(s,t), -\infty < s \leq t < \infty)$ such that for all $s \leq t$, $\Pi_{|[n]}(s,t) = \Pi^n(s,t)$. The collection $(\Pi(s,t), -\infty < s \leq t < \infty)$ is by construction a flow in the sense of Definition 3.2. Obviously, the process $(\Pi(t), t \geq 0) := (\Pi(0,t), t \geq 0)$ is a standard distinguished coalescent with coagulation measure μ.

Interpreting a classical coagulation as a reproduction and a coagulation with the distinguished block as an immigration event, a coagulation measure μ may be viewed as encoding the births and the immigration rates in some population when time goes forward.

In the same vein as Bertoin and Le Gall's flows, a population model is embedded in the dual flow, as we shall see.

3.2.3 The dual flow

Let $\hat{\mathcal{N}}$ be the image of $\mathcal{N}$ by the time reversal $t \mapsto -t$ and consider the filtration $(\hat{\mathcal{F}}_t, t \geq 0) := (\sigma(\hat{\mathcal{N}}_{[0,t]\times\mathcal{P}^0_\infty}), t \geq 0)$. For all $s \leq t$, we denote by $\hat{\Pi}(s,t)$ the partition $\Pi(-t,-s)$. The process $(\hat{\Pi}(s,t), -\infty < s \leq t < \infty)$ is called the dual flow. By a slight abuse of notation the process $(\hat{\Pi}(0,t), t \geq 0)$ will be denoted by $(\hat{\Pi}(t), t \geq 0)$. Plainly the following cocycle property holds for every $s \geq 0$,

$$\hat{\Pi}(t+s) = coag(\hat{\Pi}(t,t+s), \hat{\Pi}(t)).$$

We stress that the partition $\hat{\Pi}(t,t+s)$ is exchangeable, independent of $\hat{\mathcal{F}}_t$ and has the same law as $\hat{\Pi}(s)$. The cocycle property yields immediately that $(\hat{\Pi}(t), t \geq 0)$ and its restrictions $(\hat{\Pi}_{|[n]}(t), t \geq 0)$ are Markovian. The following property ensures that $(\hat{\Pi}(t), t \geq 0)$ is actually strongly Markovian.

Proposition 3.3 *The semigroup of the process* $(\hat{\Pi}(t), t \geq 0)$ *verifies the Feller property. For any continuous map* ϕ *from* $\mathcal{P}^0_\infty$ *to* $\mathbb{R}$*, the map* $\pi \mapsto \mathbb{E}[\phi(coag(\hat{\Pi}(t), \pi))]$ *is continuous and* $\mathbb{E}[\phi(coag(\hat{\Pi}(t), \pi))] \underset{t\to 0}{\to} \phi(\pi)$.

Proof. This is readily obtained thanks to the continuity of coagulation maps.□

Remark 6 *We stress that the process* $(\hat{\Pi}(t), t \geq 0)$ *is not a coalescent process. However, since the Poisson random measures* $\mathcal{N}$ *and* $\hat{\mathcal{N}}$ *have the same law, the process* $(\hat{\Pi}(t), t \geq 0)$ *has the same one-dimensional marginals as a standard coalescent with coagulation measure* μ.

As explained in Section 2.1, the partition-valued process $(\hat{\Pi}(t), t \geq 0)$ may be viewed as a population model forward in time. For every $t \geq s \geq 0$ and every $k \in \mathbb{N}$, we shall interpret the block $\hat{\Pi}_k(s,t)$ as the descendants at time t of the individual k living at time s and the distinguished block $\hat{\Pi}_0(s,t)$ as the descendants of the generic immigrant. Thanks to the cocycle property, for all $0 \leq s \leq t$, we have

$$\hat{\Pi}_k(t) = \bigcup_{j\in\hat{\Pi}_k(s)} \hat{\Pi}_j(s,t)$$

and the ancestor living at time s of any individual j at time t is given by $\alpha_{\hat{\Pi}(s,t)}(j)$. The random distinguished partition $\hat{\Pi}(t)$ is exchangeable and possesses asymptotic frequencies. For all $i \geq 0$, we shall interpret $|\hat{\Pi}_i(t)|$ as the fraction of the population at time t which is descendent from i. We stress that as in Donnelly-Kurtz's construction [DK99] and the generalisation [BBM$^+$09], the model is such that the higher the individual is, the faster his descendants will die. Namely, for all $t \geq 0$ and all $j \geq 0$ we have $\alpha_{\hat{\Pi}(t)}(j) \leq j$ and then for all individuals $i < j$, the descendants of i will always extinct after that of j.

Remark 7 *When neither simultaneous multiple births nor immigration is taken into account, the measure* μ *is carried on the simple partitions (meaning with only one non-trivial* *block) with a distinguished block reduced to* $\{0\}$ *and we recover exactly the "modified" look-down process of Donnelly-Kurtz for the* Λ*-Fleming-Viot process (see p195-196 of [DK99]). Moreover, the partition-valued process* $(\hat{\Pi}(t), t \geq 0)$ *corresponds in law with those induced by the dual flow of Bertoin and Le Gall* $(\hat{B}_t, t \geq 0)$ *using the paint-box scheme, see [BLG03].*

Let us study the genealogical process of this population model. We will recover a distinguished coalescent. Let $T > 0$ be a fixed time and consider the population at time T. By definition, the individuals k and l have the same ancestor at time $T - t$ if and only if k and l belong to the same block of $\hat{\Pi}(T - t, T)$. Moreover by definition of the dual flow,

$$(\hat{\Pi}(T - t, T), t \in [0, T]) = (\Pi(-T, -T + t), t \in [0, T])$$

which is a distinguished coalescent with coagulation measure μ on the interval $[0, T]$.

In the same way as Donnelly and Kurtz in [DK99], we associate initially to each individual a *type* represented by a point in a metric space $E \cup \{\partial\}$, where ∂ is an extra point not belonging to E representing the distinguished type of the immigrants. The choice of E does not matter in our setting and for the sake of simplicity, we choose for E the interval $]0, 1]$, and for distinguished type $\partial = 0$. The generic external immigrant 0 has the type $U_0 := 0$. At any time t, each individual has the type of its ancestor at time 0. In other words, for any $k \in \mathbb{N}$ the type of the individual k at time t is $U_{\alpha_{\hat{\Pi}(t)}(k)}$. The exchangeability properties of Section 2.2 will allow us to define and characterize the generalized Fleming-Viot process with immigration.

Remark 8 *We assume in this work only one source of immigration but several sources may be considered by distinguishing several blocks and types. In this work, there is no mutation assumed on the types. Birkner et al. defined in [BBM+09] a generalization of the lookdown representation and the* Ξ*-Fleming-Viot process with mutations. Assuming that no immigration or mutation is taken into account, we will recover a process with the same law as a* Ξ*-Fleming-Viot process (the measure* Ξ *is defined by* $\Xi := c_1\delta_0 + \sum_{i\geq 1} s_i^2 \nu(ds)$*). However, we stress that our method differs from that of Birkner et al.*

3.3 Generalized Fleming-Viot processes as de Finetti measures

We define and characterize in this section a measure-valued process which represents the frequencies of the types in the population at any time. This process will be called the generalized Fleming-Viot process with immigration and will be explicitly related to the *forward partition* process $(\hat{\Pi}(t), t \geq 0)$. In the same way as in [BLG03] and [BBM+09], a duality argument allows us to characterize in law the GFVIs.

3.3.1 Generalized Fleming-Viot processes with immigration

Let $\rho \in \mathcal{M}_1$, we assume that the initial types $(U_i, i \geq 1)$ are i.i.d. with law ρ and independent of $\hat{\mathcal{N}}$. For all $t \geq 0$, the random partition $\hat{\Pi}(t)$ is exchangeable and applying Lemma 3.1, we get that the sequence $(U_{\alpha_{\hat{\Pi}(t)}(l)}, l \geq 1)$ is exchangeable. We denote by ρ_t its de Finetti measure. Lemma 3.1 leads us to the following definition.

Definition 3.4 *The process $(\rho_t, t \geq 0)$ defined by*

$$\rho_t := |\hat{\Pi}_0(t)|\delta_0 + \sum_{i=1}^{\infty} |\hat{\Pi}_i(t)|\delta_{U_i} + (1 - \sum_{i=0}^{\infty} |\hat{\Pi}_i(t)|)\rho,$$

starting from $\rho_0 = \rho$, is called the generalized Fleming-Viot process with immigration.

Remark 9 *For all $t \geq 0$, the random variable ρ_t can be viewed as the Stieltjes measure of a distinguished bridge (see Chapter 1). Definition 3.4 yields a paint-box representation of the population forward in time in the same vein as the dual flow of bridges of Bertoin and Le Gall.*

Proposition 3.5 *The process $(\rho_t, t \geq 0)$ is Markovian with a Feller semigroup.*

Proof. The sequence $(U_{\alpha_{\hat{\Pi}(s+t)}(l)}, l \geq 1)$ is exchangeable with de Finetti measure ρ_{s+t}. By the cocycle property of the dual flow, we have $\hat{\Pi}(s+t) = coag(\hat{\Pi}(s, s+t), \hat{\Pi}(s))$. Therefore, for all $l \geq 1$,

$$U_{\alpha_{\hat{\Pi}(s+t)}(l)} = U_{\alpha_{\hat{\Pi}(s)} \circ \alpha_{\hat{\Pi}(s,s+t)}(l)}.$$

By Lemma 3.1, we immediately get that for all $t \geq 0$ and $s \geq 0$

$$\rho_{s+t} = (1 - \sum_{j \geq 0} |\hat{\Pi}_j(s, s+t)|)\rho_s + \sum_{j \geq 1} |\hat{\Pi}_j(s, s+t)|\delta_{U_{\alpha_{\hat{\Pi}(s)}(j)}} + |\hat{\Pi}_0(s, s+t)|\delta_0.$$

We recall that $\hat{\Pi}(s, s+t)$ is independent of $\hat{\mathcal{F}}_s$ and hence of ρ_s. By Theorem 3.4, conditionally on ρ_s, the variables $(U_{\alpha_{\hat{\Pi}(s)}(j)}, j \geq 1)$ are i.i.d. with distribution ρ_s. The process $(\rho_t, t \geq 0)$ is thus Markovian and its semigroup denoted by R_t can be described as follows. For every $\rho \in \mathcal{M}_1$, $R_t(\rho, .)$ is the law of the random probability measure

$$(1 - \sum_{i \geq 0} |\hat{\Pi}_i(t)|)\rho + \sum_{i \geq 1} |\hat{\Pi}_i(t)|\delta_{U_i} + |\hat{\Pi}_0(t)|\delta_0$$

where the variables $(U_i, i \geq 1)$ are i.i.d. distributed according to ρ and independent of $\hat{\Pi}(t)$.
We then verify that R_t enjoys the Feller property. If f is a continuous function from $\mathcal{M}_1$ to $\mathbb{R}$, the convergence in probability when $t \to 0$ of $\hat{\Pi}(t)$ to $0_{[\infty]}$ implies that $|\hat{\Pi}(t)|^{\downarrow}$ tends to 0. We then have the convergence of $R_t f$ to f when $t \to 0$. Plainly, for any sequence $(\rho^n, n \geq 1)$ which weakly converges to ρ, $R_t(\rho^n, .)$ converges to $R_t(\rho, .)$. The Feller property is then established. □
The process admits a càdlàg modification, we shall implicitly work with such a version.

3.3.2 Infinitesimal generator, core and martingale problem

As in the articles of Bertoin-Le Gall [BLG03], Donnelly-Kurtz [DK99] and Birkner et al [BBM$^+$09], the characterization in law of a GFVI will be obtained by a duality argument. Let f be a continuous function on $[0,1]^p$. Define a function from $\mathcal{M}_1 \times \mathcal{P}_p^0$ to $\mathbb{R}$ by

$$\Phi_f : (\rho, \pi) \in \mathcal{M}_1 \times \mathcal{P}_p^0 \mapsto \int_{[0,1]^{p+1}} \delta_0(dx_0)\rho(dx_1)...\rho(dx_p)f(x_{\alpha_\pi(1)}, ..., x_{\alpha_\pi(p)}).$$

Let $(\Pi(t), t \geq 0)$ be a distinguished coalescent, then we have the following lemma, where the notations $\mathbb{E}^\rho$ and $\mathbb{E}^\pi$ refer respectively to expectation when $Z_0 = \rho$ and $\Pi_{|[p]}(0) = \pi$.

Lemma 3.6

$$\mathbb{E}^{\rho}[\Phi_f(\rho_t,\pi)] = \mathbb{E}^{\pi}[\Phi_f(\rho,\Pi_{|[p]}(t))].$$

Proof of Lemma 3.6. Let $(U_i, i \geq 1)$ be independent and identically distributed with law ρ and $U_0 = 0$, we have

$$\begin{aligned}
\mathbb{E}^{\rho}[\Phi_f(\rho_t,\pi)] &= \mathbb{E}^{\rho}[\int \delta_0(dx_0)\rho_t(dx_1)...\rho_t(dx_p)f(x_{\alpha_\pi(1)},...,x_{\alpha_\pi(p)})] \\
&= \mathbb{E}[f(U_{\alpha_{\hat{\Pi}(t)}(\alpha_\pi(1))},...,U_{\alpha_{\hat{\Pi}(t)}(\alpha_\pi(p))})] \\
&= \mathbb{E}[\int \delta_0(dy_0)\rho(dy_1)...\rho(dy_p)f(y_{\alpha_{coag(\pi,\hat{\Pi}(t))}(1)},...,y_{\alpha_{coag(\pi,\hat{\Pi}(t))}(p)})] \\
&= \mathbb{E}^{\pi}[\Phi_f(\rho,\Pi_{|[p]}(t))].
\end{aligned}$$

The second equality holds because ρ_t is the de Finetti measure of $(U_{\alpha_{\hat{\Pi}(t)}(i)}, i \geq 1)$. Observing that $\alpha_{\hat{\Pi}(t)} \circ \alpha_\pi = \alpha_{coag(\pi,\hat{\Pi}(t))}$, we get the third equality. Moreover, for t fixed the random partition $\hat{\Pi}(t)$ has the same law as a standard distinguished coalescent at time t which yields the last equality.□

The Kolmogorov equations ensure that the generator $\mathcal{L}$ of the process $(\rho_t, t \geq 0)$ verifies for all continuous functions f on $[0,1]^p$, $\pi \in \mathcal{P}_p^0$ and $\rho \in \mathcal{M}_1$,

$$\mathcal{L}\Phi_f(.,\pi)(\rho) = \mathcal{L}_p^*\Phi_f(\rho,.)(\pi) \tag{3.1}$$

where $\mathcal{L}_p^*$ is the generator of the continuous time Markov chain $(\Pi_{|[p]}(t), t \geq 0)$. The process $(\rho_t, t \geq 0)$ is then characterized in law by a triplet (c_0, c_1, ν) according to the decomposition of the coagulation measure μ given in Subsection 3.2.2. Let G_f be the map defined by

$$G_f(\rho) = \int_{[0,1]^p} f(x_1,...,x_p)\rho(dx_1)...\rho(dx_p) = \Phi_f(\rho, 0_{[p]}).$$

A classical way to characterize the law of a Fleming-Viot process is to study a martingale problem, see for example Theorem 3 of [BLG03] or Proposition 1.22 of Chapter 1. The following theorem claims that a generalized Fleming-Viot process with immigration solves a well-posed martingale problem. Let f be a continuous function f on $[0,1]^p$. According to (3.1), the operator $\mathcal{L}$ is such that

$$\mathcal{L}G_f(\rho) = \sum_{\pi \in \mathcal{P}_p^0} q_\pi \int_{[0,1]^p} [f(x_{\alpha_\pi(1)},...,x_{\alpha_\pi(p)}) - f(x_1,...,x_p)]\delta_0(dx_0)\rho(dx_1)...\rho(dx_p).$$

Theorem 3.7 *The law of the process $(\rho_t, t \geq 0)$ is characterized by the following martingale problem. For every integer $p \geq 1$ and every continuous function f on $[0,1]^p$, the process*

$$G_f(\rho_t) - \textstyle\int_0^t \mathcal{L}G_f(\rho_s)ds$$

is a martingale in $(\hat{\mathcal{F}}_t, t \geq 0)$.

Remark 10 *When μ is supported on the simple distinguished partitions, we recover the well-posed martingale problem of Lemma 1.21 of Chapter 1. We then identify the processes obtained from stochastic flows of bridges in [BLG03] and Chapter 1 and from stochastic flows of partitions.*

Proof. Using (3.1), and applying Theorem 4.4.2 in [EK86], we get that there is at most one solution to the martingale problem. Dynkin's formula implies that the process in the statement is a martingale. □

The following theorem yields an explicit formula for the generator of $(\rho_t, t \geq 0)$.

Theorem 3.8 *The infinitesimal generator $\mathcal{L}$ of $(\rho_t, t \geq 0)$ verifies the following properties :*

(i) For every integer $p \geq 1$ and every continuous function f on $[0,1]^p$, we have

$$\mathcal{L}G_f = \mathcal{L}^{c_0}G_f + \mathcal{L}^{c_1}G_f + \mathcal{L}^{\nu}G_f$$

where

$$\mathcal{L}^{c_0}G_f(\rho) := c_0 \sum_{1\leq i\leq p} \int_{[0,1]^p} [f(\mathrm{x}^{0,i}) - f(\mathrm{x})]\rho^{\otimes p}(dx)$$
$$\mathcal{L}^{c_1}G_f(\rho) := c_1 \sum_{1\leq i<j\leq p} \int_{[0,1]^p} [f(\mathrm{x}^{i,j}) - f(\mathrm{x})]\rho^{\otimes p}(dx)$$
$$\mathcal{L}^{\nu}G_f(\rho) := \int_{\mathcal{P}_m^0} \{\mathbb{E}[G_f(\bar{s}\rho + s_0\delta_0 + \sum_{i\geq 1} s_i\delta_{U_i})] - G_f(\rho)\}\nu(ds)$$

where x denotes the vector $(x_1, ..., x_p)$ and

- *the vector $\mathrm{x}^{0,i}$ is defined by $\mathrm{x}_k^{0,i} = x_k$, for all $k \neq i$ and $\mathrm{x}_i^{0,i} = 0$,*
- *the vector $\mathrm{x}^{i,j}$ is defined by $\mathrm{x}_k^{i,j} = x_k$, for all $k \neq j$ and $\mathrm{x}_j^{i,j} = x_i$,*
- *the sequence $(U_i, i \geq 1)$ is i.i.d. with law ρ and $\bar{s}$ is the dust of s meaning that $\bar{s} := 1 - \sum_{i\geq 0} s_i$.*

(ii) Let $\mathcal{D}$ stand for the domain of $\mathcal{L}$. The vector space generated by functionals of the type G_f forms a core of $(\mathcal{L}, \mathcal{D})$.

Proof. (i) We have

$$\mathcal{L}G_f(\rho) = \mathcal{L}_p^*\Phi_f(\rho, .)(0_{[p]}).$$

Therefore,

$$\mathcal{L}G_f(\rho) = \sum_{\pi\in\mathcal{P}_p^0} q_\pi[\Phi_f(\rho,\pi) - \Phi_f(\rho, 0_{[p]})]$$

with $q_\pi = \mu(\mathcal{P}_{\infty,\pi}^0)$. The decomposition of μ with the triplet (c_0, c_1, ν) implies that

$$\begin{aligned}\sum_{\pi\in\mathcal{P}_p^0} q_\pi[\Phi_f(\rho,\pi) - \Phi_f(\rho, 0_{[p]})] =& c_0 \sum_{1\leq i\leq p} [f(\mathrm{x}^{0,i}) - f(\mathrm{x})]\rho^{\otimes p}(dx) + c_1 \sum_{1\leq i<j\leq p} [f(\mathrm{x}^{i,j}) - f(\mathrm{x})]\rho^{\otimes p}(dx) \\ &+ \sum_{\pi\in\mathcal{P}_p^0} \int_{\mathcal{P}_\mathbf{m}} \rho_s(\mathcal{P}_{\infty,\pi}^0)\nu(ds)[\Phi_f(\rho,\pi) - \Phi_f(\rho, 0_{[p]})].\end{aligned}$$

It remains to establish the following equality

$$\mathcal{L}^{\nu}G_f(\rho) = \sum_{\pi\in\mathcal{P}_p^0} \int_{\mathcal{P}_\mathbf{m}} \rho_s(\mathcal{P}_{\infty,\pi}^0)\nu(ds)[\Phi_f(\rho,\pi) - \Phi_f(\rho, 0_{[p]})].$$

Let $\mathbf{s} \in \mathcal{P}^0_{\mathbf{m}}$, as already mentioned, we denote by $\bar{s}$ its dust. Let $(U_i, i \geq 1)$ be i.i.d. random variables with distribution ρ. Denoting by Π an independent s-distinguished paint-box, the variables $(U_{\alpha_\Pi(j)}, j \geq 1)$ are exchangeable with a de Finetti measure which has the same law as

$$\bar{s}\rho + \sum_{i\geq 1} s_i\delta_{U_i} + s_0\delta_0.$$

Thus, we get

$$\mathbb{E}[G_f(\bar{s}\rho + \sum_{i\geq 1} s_i\delta_{U_i} + s_0\delta_0)] = \mathbb{E}[f(U_{\alpha_\Pi(1)}, ..., U_{\alpha_\Pi(p)})].$$

Moreover,

$$\mathbb{E}[f(U_{\alpha_\Pi(1)}, ..., U_{\alpha_\Pi(p)})] - \mathbb{E}[f(U_1, ..., U_p)] = \sum_{\pi\in\mathcal{P}^0_p} \mathbb{P}[\Pi_{|[p]} = \pi] \left(\mathbb{E}[f(U_{\alpha_\pi(1)}, ..., U_{\alpha_\pi(p)})] - \mathbb{E}[f(U_1, ..., U_p)]\right).$$

By definition, $\mathbb{P}[\Pi_{|[p]} = \pi] = \rho_s(\mathcal{P}^0_{\infty,\pi})$ and we get by integrating on $\mathcal{P}^0_{\mathbf{m}}$:

$$\begin{aligned}\sum_{\pi\in\mathcal{P}^0_p} \int_{\mathcal{P}_{\mathbf{m}}} \rho_s(\mathcal{P}^0_{\infty,\pi})\nu(ds) &\left(\mathbb{E}[f(U_{\alpha_\pi(1)}, ..., U_{\alpha_\pi(p)})] - \mathbb{E}[f(U_1, ..., U_p)]\right) \\ &= \int_{\mathcal{P}_{\mathbf{m}}} \mathbb{E}[f(U_{\alpha_\Pi(1)}, ..., U_{\alpha_\Pi(p)}) - f(U_1, ..., U_p)]\nu(ds) \\ &= \int_{\mathcal{P}_{\mathbf{m}}} \mathbb{E}[G_f(\bar{s}\rho + \sum_{i\geq 1} s_i\delta_{U_i} + s_0\delta_0) - G_f(\rho)]\nu(ds).\end{aligned}$$

Therefore, the statement of *(i)* is obtained.

(ii) The previous calculation yields that the map $\rho \mapsto \mathcal{L}G_f(\rho)$ is a linear combination of functionals of the type $\Phi_f(\rho, \pi)$. Besides, for all $\pi \in \mathcal{P}^0_p$, the map $\rho \mapsto \Phi_f(\rho, \pi)$ can be written as $G_g(\rho)$ with g the continuous function on $[0, 1]^{\#\pi-1}$ defined by

$$g(x_1, ..., x_{\#\pi-1}) = f(x_{\alpha_\pi(1)}, ..., x_{\alpha_\pi(p)}) \text{ with } x_0 = 0.$$

Therefore, denoting by D the vector space generated by the functionals of type G_f, the space D is invariant under the action of the generator $\mathcal{L}$. Considering the maps of the form $f(x_1, ..., x_p) = g(x_1)...g(x_p)$, we get that D contains the linear combinations of functionals $\rho \mapsto \langle g, \rho\rangle^p$. By the Stone-Weierstrass theorem, these functionals are dense in the space of continuous functions on $\mathcal{M}_1$. Thanks to the Feller property of $(\rho_t, t \geq 0)$ and according to Proposition 19.9 of [Kal02], the space D is a core. Thus, the explicit expression of $\mathcal{L}$ restricted to D, given in the statement, determines the infinitesimal generator $\mathcal{L}$ of $(\rho_t, t \geq 0)$. $\square$

3.4 Extinction of the initial types

Let $(\rho_t, t \geq 0)$ be a GFVI characterized in law by the triplet (c_0, c_1, ν). The extinction of the initial types corresponds to the absorption of $(\rho_t, t \geq 0)$ in δ_0. It means for the forward partition-valued process $(\hat{\Pi}(t), t \geq 0)$ to be absorbed at the trivial partition $(\{\mathbb{Z}_+\}, \emptyset, ...)$. We are interested in this section to determine under which conditions on (c_0, c_1, ν) the extinction occurs almost surely. We are only able to give a sufficient condition. However this condition is also necessary when the measure ν satisfies an additional assumption of regularity (as in Limic's article [Lim10]). We stress that as mentioned in Proposition 5.2 and Remark 5.3 of [BBM$^+$09], the absorption of a Ξ-Fleming-Viot process (without immigration) is closely related to the coming down from infinity of the Ξ-coalescent.

3.4.1 Extinction criterion

Define the following subspace of $\mathcal{P}_{\mathbf{m}}$

$$\mathcal{P}^f_{\mathbf{m}} := \{\mathbf{s} \in \mathcal{P}^0_{\mathbf{m}}; \sum_{i=0}^n s_i = 1 \text{ for some finite } n\}.$$

As in Section 5.5 of Schweinsberg's article [Sch00a], we consider the following cases :

- Suppose $\nu(\mathcal{P}^f_{\mathbf{m}}) = \infty$ then by a basic property of Poisson random measure, we know that $T_f := \inf\{t > 0; (t, \pi) \text{ is an atom of } \hat{\mathcal{N}} \text{ such that } \#\pi < \infty\} = 0$ almost surely. We deduce that immediately after 0, there is only a finite number of types and then the extinction occurs almost surely in a finite time.
- Suppose $0 < \nu(\mathcal{P}^f_{\mathbf{m}}) < \infty$ then T_f is exponential with parameter $\nu(\mathcal{P}^f_{\mathbf{m}})$. At time T_f only a finite number of types reproduces and extinction will occur almost surely. This allows us to reduce the problem to the case when $\nu(\mathcal{P}^f_{\mathbf{m}}) = 0$.

Suppose henceforth that $\nu(\mathcal{P}^f_{\mathbf{m}}) = 0$. We define

$$\zeta(q) := c_1 q^2/2 + \int_{\mathcal{P}^0_{\mathbf{m}}} \left(\sum_{i=1}^{\infty} (e^{-qs_i} - 1 + qs_i) \right) \nu(ds).$$

Theorem 3.9 *If the following conditions are fulfilled :*

i) $c_0 + \nu(\boldsymbol{s} \in \mathcal{P}^0_{\boldsymbol{m}}, s_0 > 0) > 0$

ii) $\int_a^\infty \frac{dq}{\zeta(q)} < \infty$ *for some* $a > 0$ *(and then automatically for all* $a > 0$*)*

then the generalized Fleming-Viot process with immigration $(\rho_t, t \geq 0)$ *is absorbed in* δ_0 *almost surely.*

Moreover, under the regularity condition (R) *:*

$$\int_{\mathcal{P}^0_m} (\sum_{i=0}^{\infty} s_i)^2 \nu(ds) < \infty,$$

the conditions i) and ii) are necessary.

Following the terminology of Limic in [Lim10], the assumption (R) is called the *regular* case.

Remark 11 *Some weaker assumption than* (R) *may be found under which the conditions i) and ii) are still necessary (see Schweinsberg's article [Sch00a] for the case of* Ξ*-coalescents). For sake of simplicity, we will not focus on that question in this article.*

Theorem 3.9 extends Theorem 1.16 in Chapter 1. Namely, the M-generalized Fleming-Viot processes always verify (R). Moreover assume that (R) holds and that the immigration and the reproduction never happen simultaneously. Therefore, the measure ν can be decomposed in the following way :

$$\nu = \nu 1_{\{\mathbf{s} \in \mathcal{P}^0_{\mathbf{m}}; \mathbf{s}=(s_0,0,\ldots)\}} + \nu 1_{\{\mathbf{s} \in \mathcal{P}^0_{\mathbf{m}}; s_0=0\}}.$$

Let us define

- $\nu_0 := \nu 1_{\{\mathbf{s} \in \mathcal{P}^0_{\mathbf{m}}; \mathbf{s}=(s_0,0,\ldots)\}}$ which is viewed as a measure on $[0,1]$, encoding immigration rate, such that $\int_0^1 s_0 \nu_0(ds_0) < \infty$.
- $\nu_1 := \nu 1_{\{\mathbf{s} \in \mathcal{P}^0_{\mathbf{m}}; s_0=0\}}$ such that $\int_{\mathcal{P}^0_{\mathbf{m}}} \left(\sum_{i \geq 1} s_i^2 \right) \nu_1(ds) < \infty$

The map ζ does not depend on ν_0. We deduce that, in this setting, the immigration has no impact on the extinction occurrence. However, we stress that the measure ν may be carried on $\{\mathbf{s} \in \mathcal{P}^0_{\mathbf{m}}; s_0 > 0, s_1 > 0\}$.

Remark 12 *If the measure ν is carried on the set $\Delta := \{(s_0, s_1) \in [0,1]^2; s_0 + s_1 < 1\}$ then the regularity assumption (R) : $\int_\Delta (s_0 + s_1)^2 \nu(ds) < \infty$ is still satisfied. Indeed for all $(s_0, s_1) \in \Delta$, $(s_0 + s_1)^2 \leq 3s_0 + s_1^2$. In this setting, Theorem 3.9 gives a necessary and sufficient condition for extinction.*

By duality, the extinction occurs if and only if there is an immigration and the embedded distinguished coalescent $(\Pi(t), t \geq 0)$ comes down from infinity, meaning that its number of blocks becomes instantaneously finite. We shall investigate this last question in the following subsection.

3.4.2 Proof of Theorem 3.9 : coming down from infinity for a distinguished coalescent

Let $(\Pi(t), t \geq 0)$ be a distinguished coalescent with triplet (c_0, c_1, ν).

Definition 3.10 *We say that a distinguished coalescent comes down from infinity if*

$$\mathbb{P}(\#\Pi(t) < \infty \text{ for all } t > 0) = 1.$$

In the same manner as Schweinsberg's article [Sch00a] Section 5.5, we will get a sufficient condition which will be necessary in a so-called *regular* case (in the same sense as in Limic's article [Lim10]). The arguments used in Chapter 1 to study the coming down for an M-coalescent may be adapted in this more general framework.

Lemma 3.11 *Consider the process $(\#\Pi(t), t \geq 0)$ of the number of blocks in a distinguished exchangeable coalescent. Provided that $\nu(\mathcal{P}^f_{\mathbf{m}}) = 0$, there are two possibilities for the evolution of the number of blocks in Π : either $\mathbb{P}(\#\Pi(t) = \infty \text{ for all } t \geq 0) = 1$ or $\mathbb{P}(\#\Pi(t) < \infty \text{ for all } t > 0) = 1$.*

Proof. This is an easy adaptation of Lemma 31 in [Sch00a]. See the Annexes. □

Let $\pi \in \mathcal{P}^0_n$, with $\#(\pi_0 \setminus \{0\}) = k_0$, $\#\pi_1 = k_1, ..., \#\pi_r = k_r$ where $r \geq 0, k_0 \geq 0$ and $k_i \geq 2$ for all $i \in [r]$ and $\sum_{i=0}^r k_i \leq n$. We will denote by $\lambda_{n,k_0,...,k_r}$ the jump rate q_π. The decomposition of the coagulation measure μ provides an explicit formula for $\lambda_{n,k_0,...,k_r}$, however its expression is rather involved and we will not use it here. We stress that by exchangeability the quantity $\lambda_{n,k_0,...,k_r}$ does not depend on the order of the integers $k_1, ..., k_r$. From a partition with n blocks, k_0-tuple, k_1-tuple,...,k_r-tuple merge simultaneously with rate $\lambda_{n,k_0,...,k_r}$. The k_0-tuple represents the blocks coagulating with the distinguished one.

Define $N(n, k_0, ..., k_r)$ to be the number of different simultaneous choices of a k_0-tuple, a k_1-tuple,.., and a k_r-tuple from a set of n elements with $k_0 \geq 0, k_i \geq 2$ for $i \in [r]$. The exact expression may be found but is not important in the rest of the current analysis. Denoting by $\Pi^*(t) = (\Pi_1(t), ...,)$, we determine the generator $G^{[n]}$ of $(\#\Pi^*_{|[n]}(t), t \geq 0)$. Let f be any map from $[n]$ to $\mathbb{R}$,

$$G^{[n]}f(l) = \sum_{r=0}^{\lfloor l/2 \rfloor} \sum_{\substack{k_0, \{k_1, ... k_r\}; \\ \sum_{i=0}^r k_i \leq l}} N(l, k_0, ..., k_r)\lambda_{l,k_0,...,k_r}[f(l - (k_0 + ... + k_r) + r) - f(l)]$$

As already mentioned, for each fixed k_0, we do not have a separate term for each possible ordering of $k_1, ..., k_r$. That is why the inner sum extends over $k_0 \geq 0$ and the multiset $\{k_1, ..., k_r\}$ such that $\sum_{i=0}^r k_i \leq l$.

We define the map Φ such that $\Phi(n)$ is the total rate of decrease in the number of blocks in $(\Pi^*(t), t \geq 0)$ when the current configuration has n blocks. We get

$$\Phi(n) = \sum_{r=0}^{\lfloor n/2 \rfloor} \sum_{\substack{k_0, \{k_1, ... k_r\}; \\ \sum_{i=0}^r k_i \leq n}} N(n, k_0, ..., k_r) \lambda_{n,k_0,...,k_r} [k_0 + ... + k_r - r].$$

Lemma 3.12 *i) A more tractable expression for Φ is given by the following*

$$\Phi(q) = c_0 q + \frac{c_1}{2} q(q-1) + \int_{\mathcal{P}_m^0} (q s_0 + \sum_{i=1}^{\infty} (q s_i - 1 + (1 - s_i)^q) \nu(ds).$$

ii) Define

$$\Psi(q) := c_0 q + c_1 q^2/2 + \int_{\mathcal{P}_m^0} \left(q s_0 + \sum_{i=1}^{\infty} (e^{-q s_i} - 1 + q s_i) \right) \nu(ds).$$

There exist C and C' two nonnegative constants such that $C\Psi(q) \leq \Phi(q) \leq C'\Psi(q)$.
iii) The map $q \mapsto \Psi(q)/q$ is concave.

Proof. Proof of i) : We have $N(q,1) = \binom{q}{1} = q$ and $N(q,0,2) = \binom{q}{2}$. Using the binomial formula, the first two terms are plain. We focus now on the integral term. Let $\mathbf{s} \in \mathcal{P}_\mathbf{m}$ and π be a s-distinguished paint-box (see Definition 5 in Chapter 1). To compute the total rate of decrease in the number of blocks from a configuration with $q \geq 1$ blocks, let us consider

$$Y_l^{(q)}(\pi) := \#\{k \in [q]; \alpha_\pi(k) = l\}.$$

Conditionally given $|\pi_l|$, the variable $Y_l^{(q)}(\pi)$ is binomial with parameters $(|\pi_l|, q)$ (degenerated in the case of $|\pi_l| = 0$) and we have

$$\begin{aligned}
\Phi(q) &= c_0 q + \frac{c_1}{2} q(q-1) + \int_{\mathcal{P}_m^0} \left(\mathbb{E}[Y_0^{(q)}(\pi)] + \sum_{l=1}^{\infty} \mathbb{E}[Y_l^{(q)}(\pi) - 1_{\{Y_l^{(q)}(\pi) > 0\}}] \right) \nu(ds) \\
&= c_0 q + \frac{c_1}{2} q(q-1) + \int_{\mathcal{P}_m^0} \left(q s_0 + \sum_{l=1}^{\infty} [q s_l - 1 + (1 - s_l)^q] \right) \nu(ds).
\end{aligned}$$

Note that these computations are exactly the same as those pages 224-225 in [Lim10] using the "coloring procedure".
Proof of ii) Same calculations as in Lemma 8 in Limic's article [Lim10].
Proof of iii) We remark that Ψ is the Laplace exponent of a spectrally positive Lévy process. Therefore the map $h : q \mapsto \Psi(q)/q$ is the Laplace exponent of a subordinator which is concave. $\square$

The following theorem may be compared with Schweinsberg's criterion in [Sch00b] and Theorem 1.16 in Chapter 1.

Theorem 3.13 *The convergence of the series $\sum_{n\geq 1}\frac{1}{\Phi(n)}$ implies the coming down from infinity of the distinguished coalescent.*

Under the regularity condition (R) :

$$\int_{\mathcal{P}^0_m}(\sum_{i=0}^{\infty}s_i)^2\nu(ds)<\infty,$$

the convergence of the series is necessary.

We refer to Example 34 p40 in [25] and Section 3.2 p231 in [20] for a coalescent which is not regular, comes down from infinity, with a divergent series. We shall follow the proof of Theorem 1.16 in Chapter 1 and study some (super)martingales.

Lemma 3.14 *Let $(\Pi(t),t\geq 0)$ be a distinguished coalescent with triplet (c_0,c_1,ν). Assume that $\nu(\mathcal{P}^f_{\boldsymbol{m}})=0$. Let us define the fixation time*

$$\zeta:=\inf\{t\geq 0:\Pi(t)=\{\mathbb{Z}_+,\emptyset,...\}\}.$$

The expectation of fixation time is bounded by

$$\mathbb{E}[\zeta]\leq\sum_{n=1}^{\infty}1/\Phi(n).$$

As a consequence, if the series in the right-hand side converges, the fixation time is finite with probability one.

Proof. Assuming the convergence of the sum $\sum_{n=1}^{\infty}1/\Phi(n)$, we define

$$f(l)=\sum_{k=l+1}^{\infty}1/\Phi(k).$$

It is easy to check directly from i) in Lemma 3.12 that the map Φ is increasing (see alternatively Lemma 28 in [Sch00a]). We thus have

$$f(l-(k_0+...+k_r)+r)-f(l)=\sum_{k=l-(k_0+...+k_r)+r+1}^{l}\frac{1}{\Phi(k)}\geq\frac{k_0+...+k_r-r}{\Phi(l)}.$$

Therefore

$$G^{[n]}f(l)\geq\sum_{r=0}^{\lfloor l/2\rfloor}\sum_{\substack{k_0,\{k_1,...,k_r\}\\ \sum_{i=0}^{r}k_i\leq l}}\lambda_{l,k_0,...,k_r}N(l,k_0,...,k_r)\frac{k_0+...+k_r-r}{\Phi(l)}=1.$$

The process $f(\#\Pi^*_{|[n]}(t))-\int_0^t G^{[n]}f(\#\Pi^*_{|[n]}(s))ds$ is a martingale. The quantity

$$\zeta_n:=\inf\{t\geq 0:\#\Pi^*_{|[n]}(t)=0\}$$

is a finite stopping time. Let $k\geq 1$, applying the optional sampling theorem to the bounded stopping time $\zeta_n\wedge k$, we get :

$$\mathbb{E}[f(\#\Pi^*_{|[n]}(\zeta_n\wedge k))]-\mathbb{E}[\int_0^{\zeta_n\wedge k}G^{[n]}f(\#\Pi^*_{|[n]}(s))ds]=f(n)$$

With the inequality $G^{[n]}f(l) \geq 1$, we deduce that

$$\mathbb{E}[\zeta_n \wedge k] \leq \mathbb{E}[f(\#\Pi^*_{|[n]}(\zeta_n \wedge k))] - f(n).$$

By monotone convergence and Lebesgue's theorem, we have $\mathbb{E}[\zeta_n] \leq f(0) - f(n)$. Passing to the limit in n, we have $\zeta_n \uparrow \zeta_\infty := \inf\{t; \#\Pi(t) = 1\}$ and $f(n) \longrightarrow 0$, thus

$$\mathbb{E}[\zeta_\infty] \leq f(0) = \sum_{k=1}^{\infty} 1/\Phi(k).$$

□

The necessary part of the proof follows exactly the steps as in Section 1.6 of Chapter 1. Assuming that (R) holds, that the coalescent comes down from infinity and that the series is infinite, we may define a supermartingale (thanks to Lemmas 3.15, 3.16 and 3.17 below) and find a contradiction by applying the optional sampling theorem (Lemma 3.18). The proofs of these lemmas are easy adaptations of those in Section refproofcdi of Chapter 1. We simply give their statements and the corresponding references.

The following technical lemma allows us to estimate the probability for the sum of n independent binomial variables to be larger than $n/2$.

Lemma 3.15 *Let $\boldsymbol{s} \in \mathcal{P}^0_{\boldsymbol{m}}$. Let π be an s-distinguished paint-box and the variables $Y_l^{(n)}(\pi)$ defined as in Lemma 3.12. For every $n_0 \geq 4$, provided that $\sum_{i=0}^{\infty} s_i$ is sufficiently small, there is the bound*

$$\mathbb{P}[\exists n \geq n_0; \sum_{l=0}^{n} Y_l^{(n)}(\pi) > \frac{n}{2}] \leq \frac{\exp\left(-n_0 f(s)\right)}{1 - \exp\left(-f(s)\right)}$$

with

$$f(s) = \frac{1}{2}\log(\frac{1}{\sum_{i\geq 0} s_i}) - \sum_{l=0}^{\infty} \log\left(\frac{s_l}{\sum_{i\geq 0} s_i} + 1 - s_l\right).$$

Proof. Easy adaptation of the arguments of Lemma 1.24 in Chapter 1.□

Lemma 3.16 *Assume that the coalescent comes down from infinity. With probability one, we have*

$$\tau := \inf\{t > 0, \#\Pi(t) < \frac{\#\Pi(t-)}{2}\} > 0.$$

Moreover, if we define $\tau_n := \inf\{t > 0, \#\Pi_{|[n]}(t) \leq \frac{\#\Pi_{|[n]}(t-)}{2}\}$, then the sequence of stopping times τ_n converges to τ almost surely.

Proof. Using the assumption (R) and the above Lemma 3.15, we get

$$\mathbb{E}[\mathcal{N}(\{(t,\pi); t \leq 1; \exists n \geq n_0, \sum_{l=0}^{n} Y_l^{(n)}(\pi) > \frac{n}{2}\})] < \infty.$$

The same arguments as in Lemma 1.25 of Chapter 1 yield the statement. □

Lemma 3.17 *Assuming (R), the coming down from infinity and that $\sum_{n\geq 1} \frac{1}{\Phi(n)} = \infty$. There exists a constant $C > 0$ such that for all $n \geq 1$, $(e^{-Ct} f(\#\Pi^*_{|[n]}(t)))_{t \leq \tau_n}$ is a non-negative supermartingale.*

Proof. With the part iii) of Lemma 3.12 and the assumption (R), this is an easy adaptation of Lemma 1.26 in Chapter 1.□

Lemma 3.18 *Under (R), if $\sum_{n\geq 1}\frac{1}{\Phi(n)}=\infty$ then Π does not come down from infinity.*

Proof. Assume that the coalescent comes down from infinity. Exactly as in Lemma 1.27 in Chapter 1, we may apply the optional sampling theorem to the previous supermartingale and find a contradiction. □

Thanks to part (ii) of Lemma 3.12, we have the following equivalence

$$\sum_{n\geq 1}\frac{1}{\Phi(n)}<\infty \Longleftrightarrow \int_a^\infty \frac{dq}{\Psi(q)}<\infty.$$

In order to establish Theorem 3.9, it suffices to show that

$$\int_a^\infty \frac{dq}{\Psi(q)}<\infty \Longrightarrow \int_a^\infty \frac{dq}{\zeta(q)}<\infty.$$

Plainly, the quantity $\frac{q}{\zeta(q)}$ is bounded. Therefore for some constants c and C,

$$\Psi(q)=\zeta(q)(1+c\frac{q}{\zeta(q)})\leq C\zeta(q).$$

Finally, we get that the conditions i) and ii) of Theorem 3.9 are necessary under the assumption (R).

Appendix

We restate and prove here Lemma 3.1.

Lemma 3.1 *Let $(U_i, i\geq 1)$ be an infinite exchangeable sequence taking values in $[0,1]$, with de Finetti measure ρ and fix $U_0=0$. Let π be an independent distinguished exchangeable partition, then the infinite sequence $(U_{\alpha_\pi(k)}, k\geq 1)$ is exchangeable. Furthermore, its de Finetti measure is*

$$(1-\sum_{i\geq 0}|\pi_i|)\rho+\sum_{i\geq 1}|\pi_i|\delta_{U_i}+|\pi_0|\delta_0.$$

Proof. By de Finetti's theorem, without loss of generality we may directly assume that the sequence $(U_i, i\geq 1)$ is i.i.d. with a distribution $\rho\in\mathcal{M}_1$. We show that for all $n\geq 1$, the random vector $(U_{\alpha_\pi(1)},...,U_{\alpha_\pi(n)})$ is then exchangeable. Let $f_1,...,f_n$ be n measurable functions on $[0,1]$ and $[n]$ be the set $\{1,...,n\}$,

$$\mathbb{E}[f_1(U_{\alpha_\pi(1)})...f_n(U_{\alpha_\pi(n)})|\pi]=\left(\prod_{i\in\pi_0\cap[n]} f_i(0)\right)\left(\prod_{k\geq 1}\int_0^1\prod_{i\in\pi_k\cap[n]} f_i(u)\rho(du)\right).$$

Let σ be a permutation of $\mathbb{Z}_+$ such that $\sigma(0) = 0$, and η be a permutation such that $\sigma^{-1}(\pi_i) = \sigma\pi_{\eta(i)}$. We stress that $\eta(0) = 0$, and we have

$$\begin{aligned}\mathbb{E}[f_1(U_{\alpha_\pi(\sigma(1))})...f_n(U_{\alpha_\pi(\sigma(n))})|\pi] &= \left(\prod_{i\in\sigma^{-1}(\pi_0)\cap[n]} f_i(0)\right)\left(\prod_{k\geq 1}\int_0^1 \prod_{i\in\sigma^{-1}(\pi_k)\cap[n]} f_i(u)\rho(du)\right)\\ &= \left(\prod_{i\in\sigma\pi_0\cap[n]} f_i(0)\right)\left(\prod_{k\geq 1}\int_0^1 \prod_{i\in\sigma\pi_{\eta(k)}\cap[n]} f_i(u)\rho(du)\right)\\ &= \left(\prod_{i\in\sigma\pi_0\cap[n]} f_i(0)\right)\left(\prod_{k\geq 1}\int_0^1 \prod_{i\in\sigma\pi_k\cap[n]} f_i(u)\rho(du)\right).\end{aligned}$$

Therefore

$$\mathbb{E}[f_1(U_{\alpha_\pi(\sigma(1))})...f_n(U_{\alpha_\pi(\sigma(n))})|\pi] = \mathbb{E}[f_1(U_{\alpha_{\sigma\pi}(1)})...f_n(U_{\alpha_{\sigma\pi}(n)})|\pi].$$

The exchangeability of the partition π ensures that

$$\mathbb{E}[f_1(U_{\alpha_{\sigma\pi}(1)})...f_n(U_{\alpha_{\sigma\pi}(n)})] = \mathbb{E}[f_1(U_{\alpha_\pi(1)})...f_n(U_{\alpha_\pi(n)})],$$

which allows us to conclude that $(U_{\alpha_\pi(1)}, ..., U_{\alpha_\pi(n)})$ is exchangeable.

We have now to identify the de Finetti measure of this exchangeable sequence. This is an easy adaptation of the proof of Lemma 4.6 of [Ber06]. For the sake of completeness, we give the proof in detail. The exchangeability is given by Lemma 3.1, and so to prove the statement it suffices, by de Finetti's theorem, to study the limit when $m \to \infty$ of

$$\frac{1}{m}\sum_{j=1}^m \delta_{U_{\alpha_\pi(j)}}.$$

We denote by S the set of singletons in the partition π. That is

$$S := \bigcup_{i\geq 0, |\pi_i|=0} \pi_i.$$

In the sequel, we will denote by S^c the complement of S. The paint-box structure of an exchangeable partition tells us that for all $t \geq 0$, S is empty or infinite. We decompose the sum in the following way

$$\begin{aligned}\frac{1}{m}\sum_{j=1}^m \delta_{U_{\alpha_\pi(j)}} &= \frac{1}{m}\sum_{j\in S, j\in[m]} \delta_{U_{\alpha_\pi(j)}} + \frac{1}{m}\sum_{j\in S^c, j\in[m]} \delta_{U_{\alpha_\pi(j)}}\\ &= \#(S\cap[m])/m\frac{1}{\#(S\cap[m])}\sum_{j\in S; j\in[m]} \delta_{U_{\alpha_\pi(j)}} + \frac{1}{m}\sum_{j\in S^c, j\in[m]} \delta_{U_{\alpha_\pi(j)}}.\end{aligned}$$

By de Finetti's theorem, we deduce that the first term converges to $|S|\rho$. Let us decompose the second term

$$\frac{1}{m}\sum_{j\in S^c, j\in[m]} \delta_{U_{\alpha_\pi(j)}} = \frac{1}{m}\sum_{i\in[m];\pi_i\subset S^c} \#(\pi_i\cap[m])\delta_{U_i}.$$

From Fatou's lemma, we get for any measurable bounded function f on $[0,1]$

$$\liminf_{m\to\infty}\sum_{i\in[m];\pi_i\subset S^c}\frac{1}{m}\#(\pi_i\cap[m])f(U_i) \geq |\pi_0|f(0) + \sum_{i\geq 1;\pi_i\subset S^c}|\pi_i|f(U_i).$$

Therefore,

$$\liminf_{m\to\infty}\langle f;\frac{1}{m}\sum_{j=1}^{m}\delta_{U_{\alpha_\pi(j)}}\rangle \geq \langle f;(1-\sum_{i\geq 0}|\pi_i|)\rho+|\pi_0|\ \delta_0+\sum_{i\geq 1}|\pi_i|\ \delta_{U_i}\rangle.$$

The last sum extends over all $i\geq 1$ because if π_i is included in S, then the quantity $|\pi_i|$ is 0.

In order to study the lim sup, define for all $\eta>0$,

$$J(\eta):=\{j\in\mathbb{Z}_+;|\pi_j|\geq\eta\}.$$

This set is finite, and we have

$$\frac{1}{m}\sum_{j\in S^c,j\in[m]}\delta_{U_{\alpha_\pi(j)}}=\frac{1}{m}\sum_{j\in J(\eta),j\in[m]}\delta_{U_{\alpha_\pi(j)}}+\frac{1}{m}\sum_{j\in S^c\setminus J(\eta),j\in[m]}\delta_{U_{\alpha_\pi(j)}}$$

The first sum extends over a finite set, so we can interchange the sum and the limit. For all $\eta>0$,

$$\frac{1}{m}\sum_{j\in J(\eta),j\in[m]}f(U_{\alpha_\pi(j)})\underset{m\to\infty}{\to}\sum_{j;\ \pi_j\subset J(\eta)}|\pi_j|f(U_j).$$

Let us study the second sum, denoting by C a constant such that $|f|\leq C$. We have for large m,

$$\frac{1}{m}\sum_{j\in S^c\setminus J(\eta),j\in[m]}f(U_{\alpha_\pi(j)})\leq C\frac{1}{m}\#(\bigcup_{j\in S^c\setminus J(\eta)}\pi_j\cap[m]).$$

When $m\to\infty$, the boundary term converges to $|\bigcup_{j\in\ S^c\setminus J(\eta)}\pi_j|$. We then have

$$\limsup_{m\to\infty}\frac{1}{m}\sum_{j\in\ S^c\cap[m]}f(U_{\alpha_\pi(j)})\leq\sum_{j;\ \pi_j\subset J(\eta)}|\pi_j|f(U_j)+C|\bigcup_{j\in S^c\setminus J(\eta)}\pi_j|.$$

By definition, we have $1-|S|-\sum_{j\in J(\eta)}|\pi_j|\underset{\eta\to 0}{\to}0$ and therefore

$$|\bigcup_{j\in S^c\setminus J(\eta)}\pi_j|\underset{\eta\to 0}{\to}0.$$

Obviously,

$$\sum_{j;\ \pi_j\subset J(\eta)}|\pi_j|f(U_j)\underset{\eta\to 0}{\to}\sum_{j\geq 0}|\pi_j|f(U_j).$$

Combining all these results, we get

$$\limsup_{m\to\infty}\langle f;\frac{1}{m}\sum_{j=1}^{m}\delta_{U_{\alpha_\pi(j)}}\rangle\leq\langle f;(1-\sum_{i\geq 0}|\pi_i|)\rho+|\pi_0|\ \delta_0+\sum_{i\geq 1}|\pi_i|\ \delta_{U_i}\rangle.$$

We then obtain the statement of the proposition.□

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Chapitre 4

Annexes

The $0-1$ law of coming down from infinity for a distinguished coalescent

The following lemma is very easy, nevertheless, this is a useful result for a good understanding of the coming down from infinity.

Lemma 4.1 *We show the following equivalence*

$$\#coag(\pi, \pi') = \infty \Longleftrightarrow \#\pi' = \#\pi = \infty.$$

Proof. The proof is very straightforward, consider the map

$$\alpha_\pi : k \mapsto \text{index of the block of } \pi \text{ containing } k$$

Since $\alpha_{coag(\pi,\pi')} = \alpha_{\pi'} \circ \alpha_\pi$, we have Im $\alpha_{coag(\pi,\pi')} =$ Im $\alpha_{\pi'} \circ \alpha_\pi$. Therefore $\#coag(\pi, \pi') = \infty$ if and only if Im $\alpha_{\pi'} \circ \alpha_\pi = \mathbb{Z}_+$. A necessary and sufficient condition for $\#coag(\pi, \pi') = \infty$ is then Im $\alpha_{\pi'} = \mathbb{Z}_+$ and Im $\alpha_\pi = \mathbb{Z}_+$; that is $\#\pi = \#\pi' = \infty$. □

Proposition 4.2 *Consider the process $(\#\Pi(t), t \geq 0)$ of the number of blocks in a distinguished exchangeable coalescent. Provided that $\mu(\{\pi; \#\pi < \infty\}) = 0$, there are two possibilities for the evolution of the number of blocks in Π :*

- *either* $\mathbb{P}(\#\Pi(t) = \infty$ *for all* $t \geq 0) = 1$
- *either* $\mathbb{P}(\#\Pi(t) < \infty$ *for all* $t > 0) = 1$.

Remark 13 *We may easily check that $\mu(\{\pi; \#\pi < \infty\}) = \nu(\mathcal{P}^f_m)$.*

Proof. We rephrase essentially the proof of Lemma 31 in [Sch00a]. Let $(\Pi(t))_{t\geq 0}$ be a distinguished coalescent with coagulation measure μ. Denote by T its time of coming down from infinity, meaning $T = \inf\{t > 0, \#\Pi(t) < \infty\}$. Assume that $\mu(\{\pi; \#\pi < \infty\}) = 0$, as requested we shall prove that a.s $T = 0$ or $T = \infty$.
By the $0-1$ Blumenthal's law, we have $\mathbb{P}[T = 0] \in \{0, 1\}$. Therefore it remains to show that $\mathbb{P}[0 < T < \infty] = 0$.

– We begin to show that

$$\mathbb{P}\left[\{\#\Pi(T-) = \infty, \#\Pi(T) < \infty, 0 < T < \infty\}\right] = 0.$$

On this event the stopping time T is a collision time, that means an atom of the underlying Poissonian measure N. By the Poissonian construction, at an atom

(t, π) of N we coagulate the partition $\Pi(t-)$ with π, we easy check that the event $\{\#\Pi(T-) = \infty, \#\Pi(T) < \infty\}$ occurs if $\Pi(T-)$ coagulates with a partition π which has a finite number of blocks. By assumption $\mu(\#\pi < \infty) = 0$, therefore this event has a zero probability.

– We study now the probability of the event $\{0 < T < \infty, \#\Pi(T-) < \infty\}$. Denote $\Pi(T-) = \{\pi_0, ..., \pi_k\}$ for some $k < \infty$. Let $n_0 = 0, n_1 = \min(\pi_1), ..., n_k = \min(\pi_k)$ and define the sequence $(T_m, m \geq 0)$ recursively as follows : $T_0 = 0$, assumed T_m defined, let $p_m \in \mathbb{N}$ such that $p_m \underset{\Pi(T_m)}{\not\sim} n_i$ for all $i \in [\,k\,]$, and define

$$T_{m+1} := \inf\{t \geq 0; p_m \underset{\Pi(t)}{\sim} n_i \text{ for some } i \in [\,k\,]\}$$

By definition, we have $T_m < T_{m+1} < ... < T$ and at each time T_m, the partition $\Pi(T_m)$ as at least $k + 2$ blocks. We then have that the quantity $T_{m+1} - T_m$ is at least exponential with parameter $\mu(\{\pi; \pi_{[\,k+1]} \neq 0_{[\,k+1]}\})$ which represents the total rate of jumps of the distinguished coalescent when it has $k + 2$ blocks. Finally, $\lim T_m = \sum_{m \geq 0}[T_{m+1} - T_m] = \infty$. We get a contradiction since for all $m \geq 0$, $T_m \leq T$.

Combining our two results, we get that $\mathbb{P}[\#\Pi(T) < \infty, 0 < T < \infty] = 0$. To conclude the proof, we show that on the event $\{T < \infty\}$, $\#\Pi(T) < \infty$ almost surely. If $\mathbb{P}[\#\Pi(T) = \infty] > 0$, by the strong Markov property at the stopping time T, conditionally on $\mathcal{F}_T$, the process $(\Pi(T+t), t \geq 0)$ has the same law as $(coag(\Pi(T), \Pi'(t)), t \geq 0)$ where $(\Pi'(t), t \geq 0)$ is an independent copy of $(\Pi(t), t \geq 0)$. The process $(\Pi(T + t), t \geq 0)$ immediately comes down from infinity. Therefore, for all $t > 0$, we have $\#coag(\Pi(T), \Pi'(t)) < \infty$ which implies $\#\Pi'(t) < \infty$. This is not possible on $\{T > 0\}$. □

The duality method

We recall here a very useful result on duality processes. The main references used are [Eth00] and [EK86]. We state here Theorem 4.4.2 of [EK86] : Let E be a separable space. Let A be an operator acting on a subspace D of $\mathcal{B}(E)$ (the bounded measurable functions of E). By a *solution of the martingale problem* for A, we mean a right continuous stochastic process $(X_t, t \geq 0)$ valued in E such that for any $f \in D$,

$$f(X_t) - \int_0^t Af(X_s)ds \text{ is a martingale with respect to the filtration } F_t^X.$$

Theorem 4.3 *Suppose that for each entrance law $\mu \in \mathcal{M}_1(E)$, any two solutions $(X_t, t \geq 0)$, $(Y_t, t \geq 0)$ of the martingale problem for A have the same one-dimensional distributions, that is for each $t > 0$,*

$$\mathbb{E}[g(X_t)] = \mathbb{E}[g(Y_t)], \text{ for } g \in \mathcal{B}(E).$$

Then, the following hold

– *there is a unique solution of the martingale problem started from μ,*
– *if A is defined on $C(E)$ and if the law of the process started at x, denoted by $\mathbb{P}_x$, satisfies $x \in E \mapsto \mathbb{P}_x(B)$ is bounded measurable for any $B \subset \mathcal{D}(\mathbb{R}_+, E)$, then the process $(X_t, t \geq 0)$ is strong Markov.*

This theorem is a basic tool for proving that martingale problems are well-posed. One approach to verify the necessary condition is *the duality method*. We give here the basic idea and refer for a complete discussion about this powerful method to Chapter 4 page 189 of Ethier-Kurtz [EK86] and Chapter 1, Section 6 of Etheridge [Eth00]. Let E_1 and E_2 be two separable metric spaces. Let A_1 and A_2 be two operators defined respectively on a subspace of $\mathcal{B}(E_1)$ and $\mathcal{B}(E_2)$. Let μ_1 and μ_2 be two initial laws on E_1 and E_2. Then the martingale problems for (A_1, μ_1), (A_2, μ_2) are said to be dual with respect to some measurable function f from $E_1 \times E_2$ to $\mathbb{R}_+$ if for each solution $(X_t, t \geq 0)$ of (A_1, μ_1) and each solution $(Y_t, t \geq 0)$ of (A_2, μ_2) (defined on the same space of probability) we have

$$\mathbb{E}_{\mu_2}[f(X_t, Y_0)] = \mathbb{E}_{\mu_1}[f(X_0, Y_t)].$$

If the set of functions $\{f(., y); y \in E_2\}$ separates the one dimensional laws of the solutions of the martingale problem associated with A_1 then the necessary condition of the theorem above is fulfilled, and the uniqueness holds. We stress that this duality argument transforms the question of uniqueness for the martingale problem (A_2, μ_2) into an existence problem for the martingale problem (A_1, μ_1).

Continuous branching processes with immigration

We gather in this section, some results about continuous branching processes with immigration.

Immigration and conditioned processes

We restate here Theorem 3.25 of [Li11].

Theorem 4.4 *Suppose $b \geq 0$ and that the Grey's condition is satisfied. Denote respectively by $(X_t, t \geq 0)$ a CB-process with reproduction mechanism Ψ and by $(Y_t, t \geq 0)$ the CBI-process with mechanisms (Ψ, Φ) with $\Phi(q) = \Psi'(q) - \Psi'(0+)$ for all $q \geq 0$. For any $t \geq 0$, and $x > 0$, the distribution of X_t under $\mathbb{P}_x(.|r + t < \zeta)$ converges as $r \to \infty$ to the law of Y_t.*

This theorem explains our claim in Chapter 2 concerning the specific cases of $c = c'$. In such case, we are actually looking at processes condition to be never extinct in the sense above. For a stronger statement, we refer to Theorem 4.1 of [Lam07] (using h-transform). For sake of completeness, here is a sketch of the proof.
Proof.

$$\begin{aligned}
\mathbb{E}_x[e^{-qX_t}|\zeta > r + t] &= \frac{\mathbb{E}_x[e^{-qX_t} 1_{\{\zeta > r+t\}}]}{\mathbb{P}_x(\zeta > r + t)} \\
&= \lim_{\theta \to \infty} \frac{\mathbb{E}_x[e^{-qX_t}(1 - e^{-\theta X_{r+t}})]}{\mathbb{E}_x[1 - e^{-\theta X_{r+t}}]} \\
&= \frac{\mathbb{E}_x[e^{-qX_t}(1 - e^{-X_t v_r(\infty)})]}{1 - e^{-v_{r+t}(\infty)}}.
\end{aligned}$$

The last one equality is obtained by applying the Markov property at time r and then taking the limit in θ. Recall that $v_{r+t}(\infty) = v_t(v_r(\infty))$. Moreover we have $v_t'(0) = e^{-bt}$ and under the Grey's condition $v = \lim_{t \to \infty} v_t(\infty) = 0$. Considering now the limit when $r \longrightarrow \infty$, we have

$$\lim_{r \to \infty} \mathbb{E}_x[e^{-qX_t}|\zeta > r + t] = \lim_{r \to \infty} \frac{\mathbb{E}_x[e^{-qX_t}(1 - e^{-X_t v_r(\infty)}) \frac{1}{v_r(\infty)}]}{\frac{1 - e^{-x v_{r+t}(\infty)}}{v_r(\infty)}}.$$

Moreover, $1 - e^{-xv_{r+t}(\infty)} \underset{r\to\infty}{\sim} v_t(v_r(\infty))x$ and thus

$$\frac{1 - e^{-xv_{r+t}(\infty)}}{v_r(\infty)} \underset{r\to\infty}{\sim} v_t'(0)x.$$

Therefore,

$$\lim_{r\to\infty} \mathbb{E}_x[e^{-qX_t}|\zeta > r+t] = \frac{1}{x}e^{bt}\mathbb{E}_x[X_t e^{-qX_t}].$$

We just have to identify the quantity on the right hand side above with the Laplace transform of Y_t. On the one hand, we have $\mathbb{E}_x[e^{-qX_t}] = e^{-xv_t(q)}$, taking the derivative with respect to q, we get

$$\mathbb{E}_x[X_t e^{qX_t}] = xe^{-xv_t(q)}\frac{\partial}{\partial q}v_t(q).$$

By an easy calculation we have :

$$\frac{\partial}{\partial t}\frac{\partial}{\partial q}v_t(q) = -\Psi'(v_t(q))v_t'(q)$$

where $v_t'(q)$ denotes $\frac{\partial}{\partial q}v_t(q)$. Furthermore

$$\frac{\partial}{\partial t}\log(v_t'(q)) = v_t'(q)^{-1}\frac{\partial}{\partial t}v_t'(q) = -\Psi'(v_t(q)).$$

We then have

$$\begin{aligned}\frac{e^{bt}}{x}\mathbb{E}_x[X_t e^{qX_t}] &= e^{bt}e^{-xv_t(q)}\exp\left[-\int_0^t \Psi'(v_s(q))ds\right]\\ &= \exp\left[-xv_t(q) - \int_0^t \Phi(v_s(q))ds\right] = \mathbb{E}_x[\exp(-qY_t)].\end{aligned}$$

□

Poissonian decomposition for continuous branching processes with immigration

In the next theorem, we provide the Poisson construction of a CBI process. This was used in Chapter 2, Section 2.3.2, Equation 2.7. We refer to [AD09] and [CR11]. Consider a CBI with

- $\Psi(q) = \frac{\sigma^2}{2}q^2 + \int_0^\infty(e^{-qx} - 1 + qx1_{\{x\le 1\}})\nu_1(dx)$ with $\sigma \ge 0$ and $\int_0^\infty(1\wedge x^2)\nu_1(dx) < \infty$,
- $\Phi(q) = \beta q + \int_0^\infty(1 - e^{-qy})\nu_0(dy)$ with $\beta \ge 0$ and $\int_0^\infty(1\wedge x)\nu_0(dx) < \infty$.

We give here a decomposition using a Poisson random measure. Let $\mathbb{P}_x$ denote the law of a CB-process started at x with mechanism $\Psi^1(q) = \int_0^\infty(e^{-qx} - 1 + qx1_{x\le 1})\nu_1(dx)$. We denote by $\mathbb{N}$ the canonical measure on the space $\mathcal{E}$ of excursions, meaning the space of paths starting at 0 and absorbed in 0. Define the measure μ on the space of càdlàg functions $\mathcal{D}(\mathbb{R}_+, \mathbb{R}_+)$ as follows

$$\mu = \beta\mathbb{N} + \int_0^\infty \nu_0(dx)\mathbb{P}_x.$$

Consider a Poisson measure $\mathcal{N} = \sum_{i\in I} \delta_{(t_i,X^i)}$ with intensity $dt \otimes \mu$. Let $(X_t, t \geq 0)$ be an independent CB-process with mechanism Ψ, then set

$$Y_t(x) = X_t(x) + \sum_{i\in I, t_i \leq t} X^i_{t-t_i}$$

Proposition 4.5 *The process $(Y_t(x), t \geq 0)$ is a CBI with mechanisms (Ψ, Φ) started at x.*

Proof. First, we have to convinve ourselves that the process $(Y_t(x), t \geq 0)$ is Markovian. We shall compute the Laplace exposant of the variable $Y_t(x)$ and find that it corresponds with that of a CBI.

$$\begin{aligned}
\mathbb{E}_x[e^{-qY_t(x)}] &= \mathbb{E}_x[e^{-qX_t(x)}]\mathbb{E}_x\left[\exp\left(-q\sum_{i\in I,t_i\leq t} X^i_{t-t_i}\right)\right] \\
&= \exp[-xv_t(q)]\exp\left[\int_0^t \mu\left[1-\exp\left(-qX_{t-s}\right)\right]ds\right] \\
&= \exp[-xv_t(q)]\exp\left[\int_0^t ds\left(\beta\mathbb{N}(1-\exp\left(-qX_{t-s}\right)) + \int_0^\infty \nu_0(dy)\mathbb{E}_y[1-\exp\left(-qX_{t-s}\right)]\right)\right] \\
&= \exp[-xv_t(q)]\exp\left[\int_0^t ds\left(\beta v_{t-s}(q) + \int_0^\infty (1-e^{-yv_{t-s}(q)})\nu_0(dy)\right)\right] \\
&= \exp[-xv_t(q)]\exp\left[-\int_0^t \left(\beta v_s(q) + \int_0^\infty (1-e^{-yv_s(q)})\nu_0(dy)\right)ds\right] \\
&= \exp\left[-xv_t(q) - \int_0^t \Phi(v_s(q))ds\right].
\end{aligned}$$

The second equality is obtained by using the exponential formula for Poisson random measure. We use after the superposition property of Poisson random measures. We deduce that the process $(Y_t, t \geq 0)$ is a CBI with mechanisms (Ψ, Φ). $\square$

Self-similarity of the stable branching processes with immigration

Let $x \geq 0$ and $\alpha \in (1, 2]$ and consider $(Y_t(x), t \geq 0)$ a CBI process with branching mechanism $\Psi(q) = dq^\alpha$ for some $d \geq 0$ and immigration mechanism $\Phi(q) = d'q^{\alpha-1}$, for some $d' \geq 0$ started at x. We establish the following self-similarity property of this process :

Proposition 4.6 *For any $x > 0$,*

$$(Y_t(x), t \geq 0) \overset{law}{=} (xY_{x^{1-\alpha}t}(1), t \geq 0)$$

Proof. Recall that

$$\mathbb{E}_x[e^{-qY_t}] = \exp\left(-xv_t(q) - \int_0^t \Phi(v_s(q))ds\right),$$

where the function v_t is the solution of the following equation $\frac{\partial}{\partial t}v_t(q) = -\Psi(v_t(q))$, $v_0(q) = q$. With the functions Φ and Ψ above, we have

$$v_t(q) = q[1 + d(\alpha-1)q^{\alpha-1}t]^{-\frac{1}{\alpha-1}}.$$

On the one hand,

$$v_{x^{1-\alpha}t}(qx) = qx[1 + d(\alpha-1)(qx)^{\alpha-1}x^{1-\alpha}t]^{-\frac{1}{\alpha-1}} = xv_t(q).$$

On the other hand, a simple calculus yields

$$\begin{aligned}
\mathbb{E}_1[e^{-qxY_{x^{1-\alpha}t}}] &= \exp\left(-v_{x^{1-\alpha}t}(qx) - d'\int_0^{x^{1-\alpha}t} v_s(qx)^{\alpha-1}ds\right)\\
&= \exp\left(-xv_t(q) - d'\int_0^t v_{x^{1-\alpha}u}(qx)^{\alpha-1}x^{1-\alpha}du\right)\\
&= \exp\left(-xv_t(q) - d'\int_0^t v_u(q)^{\alpha-1}du\right) = \mathbb{E}_x[e^{-qY_t}].
\end{aligned}$$

□

A theorem due to Volkonskiĭ on random time change and generator

Let $(E, d, \mathcal{E})$ be a measurable metric locally compact space. We consider its one point compactification by adding an external cemetery point Δ. Consider $(X_t, t \geq 0)$ a Feller process valued in E.

Theorem 4.7 (Volkonskiĭ, [Vol58]) *Let ϕ be a non-negative continuous function. Consider the following continuous additive functional $A(t) = \int_0^t \frac{ds}{\phi(X_s)}$, denote by $(T_t, t \geq 0)$ its right-continuous inverse. The quantity T_t satisfies the following equation*

$$\int_0^{T_t} \frac{ds}{\phi(X_s)} = t.$$

Define $\zeta = \inf\{t > 0; \phi(X_s) = 0\}$, and by convention set $A(t) = A(\zeta)$ for all $t \geq 0$. The time changed process $(Y_t, t \geq 0) = (X_{T_t}, t \geq 0)$ is a Feller process with lifetime $A(\zeta)$ in the time-changed filtration and its generator $\tilde{G}$ satisfies for all $f \in D$,

$$\tilde{G}f(x) = \phi(x)Gf(x).$$

Proof. We focus on the expression of the generator, we refer to Sharpe [Sha88] Chapter 8 for the regularity of the time-changed process. Since we assume E to be locally compact, both notion of extended generator and infinitesimal generator coincides (that is actually what claims the Dynkin's theorem, Theorem 5 in [Dyn56]). The process $(X_t, t \geq 0)$ is characterized in law by its infinitesimal generator G and its domain D. Denote by $\tau_\epsilon := \inf\{t > 0, d(X_t, X_0) > \epsilon\}$ for all $\epsilon > 0$. If x is not an absorbing state, then for any $f \in D$, by Dynkin's Theorem

$$Gf(x) = \lim_{\epsilon\to 0} \frac{\mathbb{E}_x[f(X_{\tau_\epsilon})] - f(x)}{\mathbb{E}_x[\tau_\epsilon]}$$

Let x such that $\phi(x) > 0$. By continuity of the function ϕ, we have

$$\mathbb{E}_x\left[\int_0^{\tau_\epsilon} \frac{ds}{\phi(X_s)}\right] \underset{\epsilon\to 0}{\sim} \frac{1}{\phi(x)}\mathbb{E}_x[\tau_\epsilon].$$

Moreover applying Dynkin's theorem for the process $(Y_t, t \geq 0)$,

$$\tilde{G}f(x) = \lim_{\epsilon\to 0} \frac{\mathbb{E}_x[f(Y_{\sigma_\epsilon})] - f(x)}{\mathbb{E}_x[\sigma_\epsilon]}$$

with $\sigma_\epsilon := \inf\{t > 0, d(Y_t, Y_0) > \epsilon\} = A(\tau_\epsilon)$. Therefore, $Y_{\sigma_\epsilon} = Y_{A(\tau_\epsilon)} = X_{\tau_\epsilon}$, and

$$\tilde{G}f(x) = \lim_{\epsilon\to 0} \frac{\mathbb{E}_x[\tau_\epsilon]}{\mathbb{E}_x[\sigma_\epsilon]} \lim_{\epsilon\to 0} \frac{\mathbb{E}_x[f(X_{\tau_\epsilon}] - f(x)}{\mathbb{E}_x[\tau_\epsilon]} = \phi(x)Gf(x)$$

The last equality holds since $\mathbb{E}_x[\sigma_\epsilon] \underset{\epsilon\to 0}{\sim} \frac{1}{\phi(x)}\mathbb{E}_x[\tau_\epsilon]$. $\square$

As an application of this theorem, we may think about the celebrate Lamperti correspondence between spectrally Lévy process and continuous branching process. Let G be the generator of a spectrally Lévy process $(X_t, t \geq 0)$ and set $\phi(x) = x$. The process $(Y_t, t \geq 0)$ has for generator $\tilde{G}f(x) = xGf(x)$ which is the generator of a CSBP process. Note that as required the state 0 is absorbing. We may show that the process $(Y_t, t \geq 0)$ has an infinite lifetime ($A(\zeta) = \infty$). Reciprocally, starting from a CSBP process $(X_t, t \geq 0)$ and setting $\phi(x) = \frac{1}{x}$, the generator $\tilde{\mathcal{G}}$ is that of a spectrally positive Lévy process. This study corresponds with that announced by John Lamperti in [Lam67a], where some results of Volkonskii are cited. We mention that Volkonskii's theorem is showed in Chapter 6 of [EK86] in a different way through martingale problems.

Random recovering of the half real line

We recall here the main result on recovering used in Chapter 2 to study the first zero of a stable CBI process. Consider a σ-finite measure μ on $\mathbb{R}_+$, denote its tail $\mu([x,\infty[)$ by $\bar{\mu}(x)$ for all $x \in \mathbb{R}_+$. Let $\mathcal{N}$ be a Poisson point process on $\mathbb{R}_+ \times \mathbb{R}_+$ with intensity $dt \otimes \mu$. Denote by $(t_i, x_i)_{i\in\mathcal{I}}$ its atoms :

$$\mathcal{N} = \sum_{i\in\mathcal{I}} \delta_{(t_i,x_i)}.$$

Recall the following covering procedure. We recover the half line by the intervals $]t_i, t_i+x_i[$, $i \in \mathcal{I}$. The set of uncovered point is

$$\mathcal{R} = [0,\infty) - \bigcup_{i\in I}]t_i, t_i + x_i[.$$

Theorem 4.8 (Fitzsimmons, Fristedt and Shepp, [FFS85]) *If*

$$\int_0^1 \exp\left(\int_t^1 \bar{\mu}(s)ds\right) dt = \infty$$

then $\mathcal{R} = \{0\}$ a.s. Otherwise $\mathcal{R}$ is the closure of the image of a subordinator with Laplace exponent Φ given by

$$\frac{1}{\Phi(q)} = \int_0^\infty e^{-qt} \exp\left(\int_t^1 \bar{\mu}(s)ds\right) dt.$$

Moreover if $\int_0^\infty \exp\left(\int_t^1 \bar{\mu}(s)ds\right) = \infty$ then the set $\mathcal{R}$ is unbounded.

Table des figures

Notations

- $\mathbb{N} := \{1, 2, ...\}$
- $\mathbb{Z}_+ := \{0, 1, 2, ...\}$
- $[n] = \{1, ..., n\}$
- $[\![n]\!] = \{0, ..., n\}$
- $\sim$ equivalence relation on $\mathbb{Z}_+$: $i \sim j$ means that i et j are in the same block of the partition.
- $\mathcal{P}_\infty$: space of partitions of $\mathbb{N}$
- $\mathcal{P}_n$: space of partitions of $[n]$
- $0_{[n]}$: partition of $[n]$ into singletons : $(\{1\}, \{2\}, ..., \{n\})$
- $\pi_{|[n]} := (\pi_1 \cap [n], \pi_2 \cap [n], ...)$
- $\mathcal{P}_\mathbf{m} = \{\mathbf{s} = (s_1, s_2, ...); s_1 \geq s_2 \geq ..., \sum_{i=1}^{\infty} s_i \leq 1\}$ (infinite simplex)
- $\mathcal{P}_\infty^0$: space of partitions of $\mathbb{Z}_+$ (called distinguished partitions).
- $\mathcal{P}_n^0$: space of partitions of $[\![n]\!]$.
- $0_{[\![n]\!]}$: partition of $[\![n]\!]$ into singletons
- $\pi_{|[\![n]\!]} := (\pi_0 \cap [\![n]\!], \pi_1 \cap [\![n]\!]], ...)$
- $\mathcal{P}_\mathbf{m}^0 := \{\mathbf{s} = (s_0, s_2, ...); s_0 \geq 0, s_1 \geq s_2 \geq ... \geq 0; \sum_{i=1}^{\infty} s_i \leq 1\}$
- $\#\pi$: number of blocks of the partition π.
- $|B| := \lim\limits_{n \to \infty} \frac{B \cap [\![n]\!]}{n}$ if the limit exists.
- $|\pi|^\downarrow := (|\pi|_0^\downarrow, |\pi|_1^\downarrow, ...)$: mass-partition of π, element of $\mathcal{P}_\mathbf{m}^0$.
- $\rho_\mathbf{s}$: law of the (distinguished) paint-box associated with the mass-partition s.
- μ : (distinguished) coagulation measure.
- $\mathbb{P}_x$: law of the canonical process started at x.
- $\mathcal{M}_1$: space of probability-measures on $[0, 1]$.
- $\mathcal{M}_f$: space of finite-measures on $[0, 1]$.
- $\langle f, \eta \rangle := \int_0^1 f(x)\eta(dx)$
- $G_f : \eta \in \mathcal{M}_f \mapsto \int_{[0,1]^m} f(x_1, ..., x_m)\eta^{\otimes m}(dx)$ with f bounded mesurable map from $[0, 1]^m$ to $\mathbb{R}$.
- $(\rho_t, t \geq 0)$: generalized Fleming-Viot process.

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